

OCT 10 1969

Courant Institute of Mathematical Sciences

Magneto-Fluid Dynamics Division

Compression of an Axially Symmetric Plasma with Non-Isotropic Pressure

Barry Marder

AEC Research and Development Report

Mathematics

February 1969



New York University

UNCLASSIFIED

New York University
Courant Institute of Mathematical Sciences
Magneto-Fluid Dynamics Division

MF-59

NYO-1480-112

COMPRESSION OF AN AXIALLY SYMMETRIC PLASMA
WITH NON-ISOTROPIC PRESSURE

Barry Marder

February 1969

U.S. Atomic Energy Commission
Contract No. AT(30-1)-1480

UNCLASSIFIED

NEW YORK UNIVERSITY
COURANT INSTITUTE-LIBRARY

Contents

	Page
Abstract	iii
Chapter I. Introduction and Summary	1
Chapter II. The equations of Motion	9
a. The two-fluid collisionless model	99
b. The ion pressure tensor	16
c. The electron pressure tensor, general case	21
d. The electron pressure tensor, theta and pure z pinches	32
e. The systems to be solved	34
f. Characteristics	39
g. An energy integral	41
h. The one-fluid model	42
Chapter III. Numerical Methods	46
a. The solution without electron pressure	46
b. The stability of the difference method without electron pressure	52
c. Numerical solution of the pinch with pressure	57
Chapter IV. Numerical Simulation of the Pinch	61
a. The theta pinch	61
b. The pure z pinch	63
c. A hybrid z-theta pinch	64
d. Other flows	68
Appendix 1. Characteristics	74
Appendix 2. The Discrete Energy Equation	79
Appendix 3. Boundary Conditions	85
Bibliography	88
Figures	90

ABSTRACT

A two-fluid collisionless model of a plasma is used to describe the behavior of a fully ionized gas under axially symmetric radial compression, a problem referred to as the pinch. Two pressure tensors are derived and used in the equations, one for the ions and one for the electrons. The ion pressure is determined by taking higher moments of the ion Vlasov equation and assuming that the third moments of the distribution function can be neglected. Expressions for the electron pressure are found from an asymptotic expansion of the electron distribution function. The expansion parameter is the square root of the ratio of the electron mass to the ion mass. The electron pressure tensor is found to depend on the magnetic fields as well as the density while the ion tensor is approximately proportional to the density raised to a power which depends on the radius.

The equations of motion are put into Lagrangian coordinates and solved numerically on the CDC 6600 digital computer. A proof of the stability of the difference method in the special case of zero electron pressure is included.

A few types of pinches are considered. The first is a theta pinch, deriving its name from the fact that the current used to compress the plasma is in the circumferential direction. In this pinch, and in the others to follow, we observe the familiar magnetic compression waves associated with the two-fluid model. A train of waves is generated at the boundary and propagates inward as the radius of the plasma column decreases. The pulses peak strongly upon reaching the center and are then reflected outward whereupon they interfere with and pass through the remaining incoming tail of the wave train. In the theta pinch the radius of the column reaches a minimum value after the applied magnetic field attains its final value. It then oscillates about a radius somewhat below its initial position.

The second pinch considered is a hybrid z-theta pinch in which a plasma which contains a constant axial magnetic field is subjected to a rising circumferential field on the boundary. As the plasma column contracts an increasing axial field is applied at the boundary in such a manner as to preserve the initial axial magnetic flux in the gas. It is found that in this way very strong compressions can be obtained without the solutions becoming discontinuous.

The last flows considered are produced by compressing a plasma which initially contains both components of the magnetic field in a force-free configuration. To initiate the collapse one component of the field is increased on the

boundary while the other is held fixed there. This results in some unusual local behavior. Depending on the initial and boundary conditions the density either rises sharply or falls strongly at the boundary as the plasma is compressed.

In all cases the compressions are sufficiently weak so that the solutions are well behaved, that is, no shocks develop. Consequently, there is considerable penetration of the magnetic fields at the boundary into the plasma.

CHAPTER I. INTRODUCTION AND SUMMARY

The study of the behavior of a plasma under cylindrical compression, the pinch problem, received its original stimulus from studies of possible methods to control and contain thermo-nuclear fusion. The problem is still under active investigation, both theoretically and experimentally. The term pinch is used to describe the motion of a plasma in either a cylindrical or toroidal configuration when an increasing magnetic field is applied to its boundary. Since, for an ionized gas, a magnetic field behaves like a pressure the gas is accelerated inward by the field. In practice the fields are produced by either running a current through the plasma or through wires surrounding the plasma. The type of current used gives the pinch its name; when the current is in the circumferential direction we speak of a theta pinch while an axial current gives rise to a z pinch.

The models used in the theoretical description of the pinch vary in complexity from the one-fluid infinite conductivity magnetohydrodynamic model represented by the Lundquist equations to self-consistent particle theories involving Vlasov equations for each component of the gas. Our treatment uses a two-fluid collision-free model which is almost consistent with the particle formulation.

Much of the research on the pinch problem has

involved numerical studies of the motion. One of the earliest was performed by Rosenbluth (19) using his so-called snow plow model in which all material swept up by the incoming boundary is piled in a thin layer and travels with the boundary. In the same paper he also investigates numerically a simplified particle model. This model predicts the formation of shocks which precede the incoming boundary and are reflected from the center.

For a more accurate description of an ionized gas a two-fluid model such as was proposed by Gardner et al. (7) is used, that is the ions and electrons are assumed to be present in two distinct gases. In a pressure-free, steady state, one dimensional solution using this model Adlam and Allen (1) found flows containing periodic oscillations or solitary waves. Their work was extended by Banos and Vernon (3) who included an isotropic fluid pressure and by Hain, Lüst, and Schlüter (11) who considered also non-isotropic pressures derived from the Vlasov equations. Gardiner and Morikawa (6) examined the effect of temperature on these solitary waves by expanding the pressure in an asymptotic series in a small parameter representing the difference between the fluid velocity and the Alfvén Mach speed. Auer, Hurwitz, and Kilb (2), Rossow (20), and Morton (15,16), have performed numerical simulations of the magnetic piston problem using two-fluid models. Morton related the time dependent

solution to the steady state results. He took the pressure to be that of an isentropic fluid with γ equal two. Although this does not give an accurate description of a warm plasma it is included to allow a qualitative description of the flow after breaking has occurred. In the case of zero pressure his model is consistent with a particle formulation. Our pressure, however, is derived from the Vlasov equations and is therefore more nearly consistent with the particle formulation for warm plasmas.

Numerical simulations of the pinch have also been performed using two-fluid models. Kilb (12), using such a representation, studied the theta pinch assuming that the electron pressure was proportional to the square of the magnetic field while ignoring the ion pressure. Hain, Hain, Roberts, Roberts, and Koppendorfer (10) included electrical and thermal conductivity in their model. They assumed that the electrons are heated ohmically while the ions are heated by shocks. The pressures of the two components are taken to be equal to the products of their respective densities and temperatures. To allow numerical computation through shocks and to determine the ion heating an artificial viscosity as discussed by Richtmyer (18) is included. Rather than relate the current to the fluid velocities they use a tensor form of Ohm's law. Following Morton, we express the current as a weighted sum of velocities of the components

so no empirical form of Ohm's law is needed. We assume the plasma is collision-free, so there is neither heat conduction nor resistivity.

Aside from the obvious effects of cylindrical convergence, the basic difference between the one dimensional piston problem and the axially symmetric two dimensional pinch problem is that since $\text{div } B$, which in cylindrical coordinates is $d(rB_{\perp})/dr$, must vanish $rB_{\perp}$ is a function only of time. For $B_{\perp}$ to be regular in the region considered it must vanish identically. Since the fields are then transverse to the flow the scale length which should be used is given by the mean gyro-magnetic radius of the ions and electrons, for this is the scale length of the transverse waves obtained by Morton. The geometry also differentiates between the two transverse fields; the influence of the axial field is quite different from that of the circumferential field. There are two main reasons for this. First, the circumferential field is required to vanish at the origin while the axial field is required only to be regular, and secondly the centrifugal effects are expressed in terms of the axial field in the equations of motion.

In this work an infinite cylinder of plasma composed of two oppositely charged fluids, the ions and electrons, is taken to be initially at equilibrium with constant density and constant axial current. This determines the pressure

and magnetic fields. For the special case in which the pressure is zero it is necessary to include a circumferential current to maintain the equilibrium. Pressure effects at the boundary are neglected so the edge appears as a sharp division between plasma and vacuum. The flows to be studied are produced by increasing the magnetic field at the boundary, compressing the plasma. The nature of the flow depends on

R_m , the initial radius of the column,

α^+ and α^- , the initial values of the ion and

electron pressures as functions of position, and

$B_2(T, Y_{\max})$ and $B_3(T, Y_{\max})$, the magnetic fields

imposed at the boundary as functions of time. Also occurring is the small parameter

ϵ , defined by $\epsilon^2 = - \frac{e^+ m^-}{e^- m^+}$, where $e^\pm$ and $m^\pm$ are

the charges and masses of the particles in the two components.

In Chapter II the equations of motion are derived from collision-free Vlasov equations for each component. The system is completed by using Maxwell's equations without displacement current, thus restricting ourselves to studying phenomena which occur on a length scale which is large compared to the Debye length. Cylindrical coordinates are used and the solution is assumed to depend only on time and radial distance. In standard fashion the first two moment equations yield conservation equations which contain mean velocities. They also contain pressure terms. To determine the ion pressure we take higher moments of the ion Vlasov

equation and assume some symmetry in the radial velocity distribution. This allows us to prove that the third moment terms are negligible. Alternatively, we could have assumed low ion temperatures to arrive at the same result. The ion pressure is found to be strongly influenced by the cylindrical geometry, with the diagonal components of the pressure tensor being nearly proportional to the density raised to a power which depends on the radius. The electron pressure is also proportional to the density, but, unlike the ion pressure, the factor of proportionality depends on the magnetic fields. The components of this tensor are found by an asymptotic expansion in ϵ of the electron distribution function in a manner suggested by Morawetz (14).

The ion and electron equations are put into Lagrangian coordinates, retaining only the lowest order terms in ϵ . Taking initial conditions into account the system is reduced to three equations for R , B_2 and B_3 where R is the radius expressed as a function of the new coordinates and B_2 and B_3 are the circumferential and axial magnetic fields. The equations are of mixed hyperbolic-parabolic type with the only non-trivial characteristics being the forward and backward sound speeds.

The one-fluid analog to the problem is briefly examined. It is shown that, with proper scaling of the independent variables, the one-fluid solution is obtained from the two-fluid one by a singular limiting process in

which the scaling parameter goes to zero.

Chapter III contains the numerical methods used to solve the equations derived in the preceding chapter. A proof of the stability of the method used is given for the special case in which the electron pressure is absent and the ion pressure is initially constant. The proof depends on the energy method and uses an energy integral of the system.

The last chapter contains the results of the numerical simulation of the two types of pinch. For the theta pinch we observe the characteristic two-fluid compression waves being generated at the outer boundary and propagating inward with a speed somewhat in excess of the Alfvén-sound speed predicted by the linearized theory. The speed depends on the amplitude of the waves which can become quite large without initiating the formation of shocks. When the disturbance reaches the center the density rises very strongly. As each wave arrives at the center it is reflected and travels back through the plasma interacting with the remaining tail of the wave train.

We investigate also a hybrid z-theta pinch in which a plasma containing a constant axial magnetic field at equilibrium is compressed by imposing an increasing circumferential field at the boundary. We impose also the condition that B_z vanishes at the boundary. This is almost equivalent to the requirement that the axial magnetic flux in the plasma

is conserved. With these conditions we find that very strong compressions with large density rises are produced. What is more, the solution remains well behaved for a time far longer than one would expect from the results of the theta pinch calculations.

Finally, we look at some flows which, while they might not be experimentally realistic, exhibit some rather unusual phenomena. In these flows we have both magnetic fields present initially in a force free configuration. By specifying that the initial axial current is constant the state is uniquely determined. We then raise one of the fields at the boundary and require that the other field is held fixed there at its initial value. By imposing this condition we observe some paradoxical effects. When B_2 is raised the plasma develops a sharp rise in density at the boundary while, for a particular initial configuration, raising B_3 produces the opposite effect, the density at the boundary falls. These results are explained by the effects of the initial field geometry and the special nature of the boundary conditions. The compression waves produced in these cases are similar to, but weaker than, those observed in the theta pinch.

ACKNOWLEDGEMENTS: I wish to thank Professors C. S. Morawetz, H. Weitzner, and H. Grad for their assistance in this work.

CHAPTER II. THE EQUATIONS OF MOTION

In this chapter we discuss the model used to represent the plasma and derive expressions for the pressure tensors which appear in the equations of motion. Cylindrical Lagrangian coordinates are used to put the equations into a form suitable for numerical computation.

a. The two-fluid collisionless model

We consider a model in which the plasma is composed of two oppositely charged components; the dynamics of each being governed by collisionless Vlasov equations for their distribution functions $f^\pm(t, \bar{r}, \bar{u})$. The plus and minus superscripts denote ion and electron terms. The arguments of these distribution functions are time, the position vector, and the velocity vector. The electric and magnetic fields, $\bar{E}(t, \bar{r})$ and $\bar{B}(t, \bar{r})$, and the current $\bar{J}(t, \bar{r})$ obey pre-Maxwell equations. The neglect of the displacement current restricts us to studying low frequency phenomena. We have then, in rationalized mks units,

$$(2.1) \quad f_t^\pm + \bar{u} \cdot \bar{\nabla}_r f^\pm + \bar{F}^\pm \cdot \bar{\nabla}_u f^\pm = 0$$

$$(2.2) \quad \left\{ \begin{array}{l} \bar{\nabla}_r \cdot \bar{B} = 0 \\ \bar{\nabla}_r \times \bar{B} = \mu \bar{J} \\ \bar{\nabla}_r \times \bar{E} = - \bar{B}_t \\ \kappa \bar{\nabla}_r \cdot \bar{E} = q \end{array} \right.$$

$$(2.3) \quad \begin{aligned} \bar{J} &= \int \bar{u} (e^+ r^+ - e^- r^-) |d\bar{u}| \\ q &= \int (e^+ r^+ + e^- r^-) |d\bar{u}| \end{aligned}$$

where:

$\bar{F}^\pm$ is the force per unit mass on an ion or electron traveling with velocity $\bar{u}$,

q is the charge density,

μ is the permeability,

ϵ is the dielectric constant, and

$e^\pm$ is the charge on a particle.

By assuming that all relevant length scales are large compared to the Debye length we can substitute the approximation $q \approx 0$ for Poisson's equation in (2.2).

To put the equations into dimensionless form let $\bar{B}_0$ and N_0 be some suitable initial values of the magnetic field and the total number density. Define

$$a_0 = \sqrt{B_0^2 / \mu N_0 (m^+ + m^-)} \quad , \quad l_0 = \sqrt{-m^+ m^- / e^+ e^- \mu N_0 (m^+ + m^-)}$$

where $m^\pm$ is the particle mass. a_0 is the initial Alfvén velocity and l_0 is the geometric mean of the gyromagnetic radii of the ions and electrons moving across a field $\bar{B}_0$ with speed a_0 . These quantities are used to normalize the variables in the following manner:

$$(2.4) \quad \begin{aligned} r: & \quad l_0 & \bar{E}: & \quad a_0 |B_0| \\ t: & \quad l_0 / a_0 & \bar{B}: & \quad |B_0| \\ N: & \quad N_0 & \bar{u}: & \quad a_0 \end{aligned}$$

The new dimensionless variables will be denoted by the same symbols.

Since the pinch problem deals with the radial compression of the plasma it is natural to assume that the solution is axially symmetric, being dependent only on the radius "r" and time "t". Let the subscripts 1, 2 and 3 denote the r, θ and z components of a vector. The force terms can then be written

$$\begin{aligned}
 F_1^+ &= K\varepsilon^K(E_1 + u_2 B_3 - u_3 B_2) + \frac{1}{r} u_2^2 \\
 (2.5) \quad F_2^+ &= K\varepsilon^K(E_2 + u_3 B_1 - u_1 B_3) - \frac{1}{r} u_1 u_2 \\
 F_3^+ &= K\varepsilon^K(E_3 + u_1 B_2 - u_2 B_1)
 \end{aligned}$$

where

$$K = \begin{cases} 1 & \text{for the + equation} \\ -1 & \text{for the - equation} \end{cases} \quad \text{and} \quad \varepsilon^2 = - \frac{e^+ m^-}{e^- m^+} .$$

The force components are seen to be composed of $\bar{E} + \bar{u} \times \bar{B}$ terms plus centrifugal effects arising from the cylindrical geometry. For singly ionized particles ε^2 is the mass ratio.

The first equation of (2.2) is $(rB_1)_r = 0$. To avoid the existence of a magnetic monopole at the origin B_1 is taken to be identically zero. Thus, using the force terms from (2.5) the Vlasov equations take the form

$$\begin{aligned}
 (2.6) \quad f_t^+ + u_1 f_r^+ + \frac{u_2^2}{r} f^+ u_1 - \frac{u_1 u_2}{r} f^+ u_2 + K\varepsilon^K[(E_1 + u_2 B_3 - u_3 B_2) f^+ u_1 \\
 + (E_2 - u_1 B_3) f^+ u_2 + (E_3 + u_1 B_2) f^+ u_3] = 0
 \end{aligned}$$

The macroscopic equations of motion are obtained from (2.6) by taking moments. We define first the ion and electron number densities and their mean velocities:

$$N^{\pm} = \int f^{\pm} |d\bar{u}| \quad , \quad (U^{\pm}, V^{\pm}, W^{\pm}) = \int \bar{u} f^{\pm} |d\bar{u}| \bigg/ N^{\pm}$$

Direct integration of (2.6) gives the conservation of mass equations:

$$(2.7) \quad (rN^{\pm})_t + (rN^{\pm}U^{\pm})_r = 0$$

The assumption of charge neutrality gives $N^{+} = N^{-}$, so subtracting the minus from the plus equation in (2.7) yields $(r(U^{+} - U^{-}))_r = 0$. Again, to avoid singularities at the origin, we must have $U^{+} = U^{-}$. We denote these common values by N and U . Momentum conservation equations are obtained by multiplying equations (2.6) by u_1 , u_2 and u_3 and integrating, assuming that the distribution functions vanish sufficiently rapidly as the velocity goes to infinity. This gives

$$(2.8) \quad \begin{aligned} & (rNU)_t + (rNU^2)_r - NV^{\pm 2} + \frac{1+\varepsilon^2}{\varepsilon^{1-K}} [(rP_{11}^{\pm})_r - P_{22}^{\pm}] \\ & \quad - K\varepsilon^K Nr[E_1 + V^{\pm}B_3 - W^{\pm}B_2] = 0 \\ & (rNV^{\pm})_t + (rNUV^{\pm})_r + NUV^{\pm} + \frac{1+\varepsilon^2}{\varepsilon^{1-K}} [(rP_{12}^{\pm})_r + P_{12}^{\pm}] \\ & \quad - K\varepsilon^K Nr[E_2 - UB_3] = 0 \\ & (rNW^{\pm})_t + (rNUW^{\pm})_r + \frac{1+\varepsilon^2}{\varepsilon^{1-K}} (rP_{13}^{\pm})_r \\ & \quad - K\varepsilon^K Nr[E_3 + UB_2] = 0 \end{aligned}$$

where $P_{ij}^{\pm} = \frac{\varepsilon^{1-K}}{1+\varepsilon^2} \int [u_i - \int u_i f^{\pm} |d\bar{u}|] [u_j - \int u_j f^{\pm} |d\bar{u}|] f^{\pm} |d\bar{u}|$.

Equations (2.2) take the form

$$\begin{aligned}
 B_{2t} &= E_{3r} , & rB_{3t} &= - (rE_2)_r \\
 (2.9) \quad (rB_2)_r &= rJ_3 = \frac{\varepsilon}{1+\varepsilon^2} rN(W^+ - W^-) \\
 B_{3r} &= - J_2 = \frac{-\varepsilon}{1+\varepsilon^2} N(V^+ - V^-)
 \end{aligned}$$

The system (2.7), (2.8) and (2.9) contains eleven equations for the nineteen quantities N , U , $V^\pm$, $W^\pm$, B_2 , B_3 , E_1 , E_2 , E_3 , $P_{11}^\pm$, $P_{22}^\pm$, $P_{12}^\pm$, $P_{13}^\pm$. To close the system it is necessary to obtain expressions for the eight pressure terms.

It is useful at this point to introduce Lagrangian coordinates which simplify the equations and have the desirable effect of leaving the boundary of the plasma fixed in the new coordinates. Let Y be defined for each point in the plasma as $1/(2\pi)$ times the total number of ions lying between the origin and the point and let T be the total time derivative. Thus, if we let $R(T, Y) = r$, we have

$$Y = \int_0^{R(T, Y)} rN(t, r) dr , \quad \frac{dY}{dT} = 0 .$$

The transformation to Lagrangian coordinates can be expressed in the more useful differential form as

$$\frac{\partial}{\partial Y} = \frac{1}{R\dot{R}} \frac{\partial}{\partial r} , \quad \frac{\partial}{\partial T} = \frac{\partial}{\partial t} + U \frac{\partial}{\partial r} ,$$

from which we readily see that

$$N = \frac{1}{R\dot{R}_Y} , \quad U = R_{\dot{T}} .$$

Equation (2.7) becomes an identity in the new coordinates and (2.8) and (2.9) take the form

$$(2.10) \left\{ \begin{aligned} R_{TT} - \frac{V^{\pm 2}}{R} + \frac{1+\varepsilon^2}{\varepsilon^{1-K}} [(RP_{11}^{\pm})_Y - R_Y P_{22}^{\pm}] - K\varepsilon^K [E_1 + V^{\pm} B_3 - W^{\pm} B_2] &= 0 \\ (RV^{\pm})_T + \frac{1+\varepsilon^2}{\varepsilon^{1-K}} (R^2 P_{12}^{\pm})_Y - K\varepsilon^K R [E_2 - R_T B_3] &= 0 \\ W_T^{\pm} + \frac{1+\varepsilon^2}{\varepsilon^{1-K}} (R P_{13}^{\pm})_Y - K\varepsilon^K [E_3 + R_T B_2] &= 0 \end{aligned} \right.$$

$$(2.11) \left\{ \begin{aligned} R_Y B_{2T} - R_T B_{2Y} &= E_{3Y} \\ RR_Y B_{3T} - RR_T B_{3Y} &= - (RE_2)_Y \\ (RB_2)_Y &= \frac{\varepsilon}{1+\varepsilon^2} (W^+ - W^-) \\ (RB_3)_Y &= \frac{-\varepsilon}{1+\varepsilon^2} (V^+ - V^-) \end{aligned} \right.$$

This system allows considerable simplification.

The electric fields in (2.10) are eliminated by multiplying the electron equations by ε^2 and adding them to the corresponding ion equations. This gives

$$(2.12) \left\{ \begin{aligned} (1+\varepsilon^2)R_{TT} - \frac{1}{R} (V^{+2} + \varepsilon^2 V^{-2}) + (1+\varepsilon^2)[(R(P_{11}^+ + P_{11}^-))_Y \\ - R_Y (P_{22}^+ + P_{22}^-)] - \varepsilon B_3 (V^+ - V^-) + \varepsilon B_2 (W^+ - W^-) &= 0 \\ (R(V^+ + \varepsilon^2 V^-))_T + (1+\varepsilon^2)(R^2 (P_{12}^+ - P_{12}^-))_Y &= 0 \\ (W^+ + \varepsilon^2 W^-)_T + (1+\varepsilon^2)(R(P_{13}^+ + P_{13}^-))_Y &= 0 \end{aligned} \right.$$

To eliminate the electric fields from the remaining equations we subtract the electron equations from their ion counterparts

in the second and third members of (2.10). The results are differentiated with respect to Y to yield

$$(R(V^+ - V^-))_{TY} + (R^2(P_{12}^+ - \frac{1}{\epsilon^2} P_{12}^-))_{YY} - \frac{1+\epsilon^2}{\epsilon} ((RE_2)_Y - (RR_TB_3)_Y) = 0$$

$$(W^+ - W^-)_{TY} + (R(P_{13}^+ - \frac{1}{\epsilon^2} P_{13}^-))_{YY} - \frac{1+\epsilon^2}{\epsilon} (E_{3Y} + R_TB_2)_Y = 0$$

Using the equations in (2.11) we obtain the following equations for the magnetic fields:

$$[(RB_2)_{YY} - R_Y B_2]_T + \frac{\epsilon}{1+\epsilon^2} [R(P_{13}^+ - \frac{1}{\epsilon^2} P_{13}^-)]_{YY} = 0 \quad (2.13)$$

$$[(R^2 B_{3Y})_Y - RR_Y B_3]_T - \frac{\epsilon}{1+\epsilon^2} [R^2(P_{12}^+ - \frac{1}{\epsilon^2} P_{12}^-)]_{YY} = 0$$

The first equation in (2.12) can be simplified using (2.11) and the identity

$$V^{+2} + \epsilon^2 V^{-2} \equiv \frac{1}{1+\epsilon^2} [(V^+ + \epsilon^2 V^-)^2 + \epsilon^2 (V^+ - V^-)^2].$$

It then becomes

$$(2.14) \quad R_{TT} - R(B_{3Y})^2 + B_2(RB_2)_Y + RB_3 B_{3Y} + (R(P_{11}^+ + P_{11}^-))_Y - R_Y(P_{22}^+ + P_{22}^-) = \frac{(V^+ + \epsilon^2 V^-)^2}{R(1+\epsilon^2)^2}$$

We can proceed no further without finding explicit representations for the pressure terms. It should be noted here that if we assume that all pressure terms are zero, the second equation in (2.12) shows that the right hand side of (2.14) can be taken to be identically zero. We have then

in (2.13) and (2.14) three equations for R , B_2 and B_3 .

Since both the ion and electron pressure terms will be determined using series developments in the small parameter ε , it is necessary to know the order of all quantities appearing in the equations. R , B_2 and B_3 are of order one since they were used to define the scaling parameters. From (2.11) we find that E_2 and E_3 are also of this order. The first equation in (2.10) in which E_1 appears with R_{TT} leads us to conjecture that E_1 is $O(1/\varepsilon)$. We will use this in determining the orders of the remaining terms and we shall see that this conjecture is consistent with the solution. Define $E = \varepsilon E_1$. E is $O(1)$ and is assumed to be regular.

b. The ion pressure tensor

To obtain equations for the components of the ion pressure tensor we make the assumption that the ion distribution function is symmetric about the mean radial velocity, that is, $f^\dagger(t, r, U+u_1, u_2, u_3) = f^\dagger(t, r, U-u_1, u_2, u_3)$. Grad (8) and Chew, Goldberger, and Low (4) have made similar assumptions in cases where the rapid spiraling of the particles about a guiding center implies isotropy of the stress tensor in the plane perpendicular to the magnetic field. This argument fails, however, for the relatively heavy ions. Instead, we assume symmetry in the radial direction and show that this assumption implies symmetry to first order

in ε in the other directions.

To prove that the distribution function is symmetric in the θ and z components of velocity to first order in ε we define two functions G and H by

$$G(t, r, u_1, u_2, u_3) = f^+(t, r, u_1, u_2, u_3) - f^+(t, r, u_1, -u_2, u_3) \quad \text{and}$$

$$H(t, r, u_1, u_2, u_3) = f^+(t, r, u_1, u_2, u_3) - f^+(t, r, u_1, u_2, -u_3) .$$

G and H measure the symmetry of the ion distribution function in the planes perpendicular to the u_2 and u_3 axes. From (2.6) we find differential equations satisfied by these functions:

$$G_t + u_1 G_r + \frac{u_2^2}{r} G_{u_1} + E G_{u_1} - \frac{u_1 u_2}{r} G_{u_2} = O(\varepsilon)$$

$$H_t + u_1 H_r + \frac{u_2^2}{r} H_{u_1} + E H_{u_1} - \frac{u_1 u_2}{r} H_{u_2} = O(\varepsilon) .$$

If we take G and H to be initially zero then it is clear that they will remain $O(\varepsilon)$ for all finite time. Thus, the ion distribution function is symmetric to $O(\varepsilon)$ about the values $u_2 = 0$ and $u_3 = 0$. We have then the desired result that all third moments vanish to this order.

There is another justification for ignoring third moments which involves a temperature related parameter.

We define $U_0^+(t, r)$, $V_0^+(t, r)$ and $W_0^+(t, r)$ by

$$U_{0_t}^+ + U_0^+ U_{0_r}^+ + \frac{V_0^{+2}}{r} + E = 0 , \quad (r V_0^+)_t + U_0^+ (r V_0^+)_r = 0 ,$$

$$W_{0_t}^+ + U_0^+ W_{0_r}^+ = 0 ,$$

and p , q and s by

$$u_1 - U_0^+ = \sigma p, \quad u_2 - V_0^+ = \sigma q, \quad u_3 - W_0^+ = \sigma s$$

where σ is a parameter related to the temperature. We write (2.6) in the form

$$f_t^+ + u_1 f_r^+ + \frac{u_2^2}{r} f^+ u_1 - \frac{u_1 u_2}{r} f^+ u_2 + E f^+ u_1 = O(\varepsilon)$$

and substitute into this the new variables p, q and s giving

$$\begin{aligned} \sigma^{-1} [(U_{0t}^+ + U_0^+ U_{0r}^+ + \frac{V_0^{+2}}{r} + E) \hat{f}_p - \frac{1}{r} (r V_{0t}^+ + U_0^+ (r V_0^+)_r) \hat{f}_q] \\ + \hat{f}_t + U_0^+ \hat{f}_r + 2 \frac{q V_0^+}{r} \hat{f}_p - \frac{1}{r} (q U_0^+ + p V_0^+) \hat{f}_q \\ + \sigma [p \hat{f}_r + \frac{q^2}{r} \hat{f}_p - \frac{pq}{r} \hat{f}_q] = O(\varepsilon), \end{aligned}$$

where $\hat{f}(t, r, p, q, s) = f(t, r, u_1, u_2, u_3)$. If we now consider σ a small parameter and express $\hat{f}$ as a power series in it, we would find that the zero order term satisfies

$$\hat{f}_{0t} + U_0^+ \hat{f}_r + 2 \frac{q V_0^+}{r} \hat{f}_p - \frac{1}{r} (q U_0^+ + p V_0^+) \hat{f}_q = O(\varepsilon).$$

Using this we find that all third moments are $O(\sigma^3)$ and can be neglected. The condition that σ is small is that the temperature of the ions is low.

Rather than introduce new parameters we shall consider the third moments to be $O(\varepsilon)$ as indicated previously.

We now take higher moments of the ion distribution function and express these in Lagrangian coordinates. The computations are straightforward and will be omitted. Upon multiplication of (2.6) by $u_1, u_1^2, u_2^2, u_3^2, u_2, u_3, u_1 u_2$, and

$u_1 u_3$ we obtain

$$(2.15) \left\{ \begin{aligned} & R_{TT} - \frac{V^{+2}}{R} + (RP_{11}^+)_{\text{Y}} - R_Y P_{22}^+ - E = O(\varepsilon) \\ & (R_T^2)_T - \frac{2}{R} R_T V^{+2} + R_Y (RP_{11}^+)_{\text{T}} + \frac{1}{RR_T^2 P_{11}^+} (R_T^3 R^2 P_{11}^{+2})_{\text{Y}} \\ & \quad - 4 V^+ R_Y P_{12}^+ - 2 R_Y R_T P_{22}^+ - 2 R_T E = O(\varepsilon) \\ & (V^{+2})_{\text{T}} + \frac{2}{R} R_T V^{+2} + RR_{TY} P_{22}^+ + R_Y (RP_{22}^+)_{\text{T}} + \frac{2}{R^2} (R^3 V^+ P_{12}^+)_{\text{Y}} \\ & \quad + 2 R_T R_Y P_{22}^+ = O(\varepsilon) \\ & (W^{+2})_{\text{T}} + (RR_Y P_{33}^+)_{\text{T}} + 2(RW^+ P_{12}^+)_{\text{Y}} = O(\varepsilon) \end{aligned} \right.$$

$$(2.16) \left\{ \begin{aligned} & (RV^+)_{\text{T}} + (R^2 P_{12}^+)_{\text{Y}} = O(\varepsilon) \\ & W_T^+ + (R P_{13}^+)_{\text{Y}} = O(\varepsilon) \\ & (V^+ R_T)_{\text{T}} + \frac{1}{R} V^+ (R_T)^2 + R_Y (R P_{12}^+)_{\text{T}} + \frac{1}{R^2 R_T} (R^3 R_T^2 P_{12}^+)_{\text{Y}} \\ & \quad + (V^+ R P_{11}^+)_{\text{Y}} + V^+ R_Y P_{11}^+ - EV^+ = O(\varepsilon) \\ & (W^+ R_T)_{\text{T}} + R_Y (RP_{13}^+)_{\text{T}} + \frac{1}{R^2 R_T P_{13}^{+2}} (R_T^2 R^3 P_{13}^{+3})_{\text{Y}} \\ & \quad + (RW^+ P_{11}^+)_{\text{Y}} - EW^+ = O(\varepsilon) \end{aligned} \right.$$

The right hand sides of (2.15) and (2.16) are composed of third moments and terms multiplied by ε . Notice that the first equation shows that E is indeed of order one. The four equations of (2.16) show that if V^+ , W^+ , P_{12}^+ , and P_{13}^+ are initially zero they remain $O(\varepsilon)$ for all time, a fact which we already knew from (2.14). Discarding these terms from the left hand sides of (2.15) we are able to eliminate E from the remaining equations. We have then

$$\begin{aligned}
& (RR_Y^3 P_{11}^+)_{\text{T}} = O(\epsilon) , \quad (R^3 R_Y P_{22}^+)_{\text{T}} = O(\epsilon) , \\
(2.17) \quad & (RR_Y P_{33}^+)_{\text{T}} = O(\epsilon) , \\
& P_{12}^+ = O(\epsilon) , \quad P_{13}^+ = O(\epsilon) , \quad P_{23}^+ = O(\epsilon) .
\end{aligned}$$

The ion pressure terms as determined by (2.17) contain three arbitrary functions of Y which must be determined from initial conditions. Since $N = 1/RR_Y$ we can write

$$P_{11}^+ = R^2 N^3 h_1(Y) , \quad P_{22}^+ = \frac{N}{R^2} h_2(Y) , \quad P_{33}^+ = N h_3(Y) .$$

We shall take as an initial condition $N = 1$ which gives $R^2 = 2Y$. Thus, if we let $\alpha_0^+(Y)$ be the initial value of the diagonal terms in the ion pressure tensor we find $h_1(Y) = \alpha^+/2Y$, $h_2(Y) = 2\alpha^+Y$, and $h_3(Y) = \alpha^+$ leaving

$$(2.18) \quad P_{11}^+ = \alpha^+/2YRR_Y^3 , \quad P_{22}^+ = 2\alpha^+Y/R^3 R_Y , \quad P_{33}^+ = \alpha^+/RR_Y .$$

The form of the ion pressure tensor is unusual. If we let $N^* = \int_0^R r N dr / \int_0^R r dr$, that is, the average number density of the ions between the center and R , we can express (2.18) in the form $P_{11}^+ = \alpha^+ N^3 / N^*$, $P_{22}^+ = \alpha^+ N N^*$, $P_{33}^+ = \alpha^+ N$. Near the center N^* will not differ significantly from N , assuming the fluctuations in density are not too violent. Here then, both P_{11}^+ and P_{22}^+ are nearly proportional to N^2 . Far from the center, however, N^* is nearly constant, depending little on local variations in density. P_{11}^+ now behaves like N^3 while P_{22}^+ is like N . P_{33}^+ , being less

influenced by the cylindrical geometry is everywhere proportional to N.

c. The electron pressure tensor, general case

To determine expressions for the electron pressure we find an asymptotic expansion of the electron distribution function and from this determine the pressure tensor. Our method, suggested by Gardner, is similar to that used in guiding center theory but differs in that E_1 is $O(1/\epsilon)$.

Following Morawetz (14) we write the electron Vlasov equation in the form

$$(2.19) \quad \begin{aligned} f_t^- + u_1 f_r^- + \frac{1}{\epsilon^2} \left(\frac{\epsilon^2 u_2^2}{r} - \epsilon E_1 - \epsilon u_2 B_3 + \epsilon u_3 B_2 \right) f_{u_1}^- \\ - \frac{1}{\epsilon} \left(\frac{\epsilon u_1 u_2}{r} + E_2 - u_1 B_3 \right) f_{u_2}^- - \frac{1}{\epsilon} (E_3 + u_1 B_2) f_{u_3}^- = 0. \end{aligned}$$

From (2.19) we find that f^- will be constant along particle paths given by

$$\begin{aligned} \frac{du_1}{dt} &= u_{1t} + u_1 u_{1r} = \frac{1}{\epsilon^2} \left(\frac{\epsilon^2 u_2^2}{r} - \epsilon E_1 - \epsilon u_2 B_3 + \epsilon u_3 B_2 \right) \\ \frac{du_2}{dt} &= u_{2t} + u_1 u_{2r} = - \frac{1}{\epsilon} \left(\frac{\epsilon u_1 u_2}{r} + E_2 - u_1 B_3 \right) \\ \frac{du_3}{dt} &= u_{3t} + u_1 u_{3r} = - \frac{1}{\epsilon} (E_3 + u_1 B_2) \end{aligned}$$

These equations suggest looking for series representations of the velocities in the form

$$\begin{aligned}
U &= U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \dots \\
(2.21) \quad V^- &= \frac{1}{\varepsilon} V_{-1} + V_0 + \varepsilon V_1 + \dots \\
W^- &= \frac{1}{\varepsilon} W_{-1} + W_0 + \varepsilon W_1 + \dots
\end{aligned}$$

where, motivated by (2.19) we define

$$\begin{aligned}
(2.22) \quad & \frac{1}{r} V_{-1}^2 - V_{-1} B_3 + W_{-1} B_2 - E = 0 \\
& V_{-1t} + U_0 V_{-1r} = - \left(-\frac{U_0 V_{-1}}{r} + E_2 - U_0 B_3 \right) \\
& W_{-1t} + U_0 W_{-1r} = - (E_3 + U_0 B_2) .
\end{aligned}$$

We define new velocity variables ξ , η , and ζ by $\xi = u_1 - U_0$, $\eta = u_2 - \frac{1}{\varepsilon} V_{-1}$, and $\zeta = u_3 - \frac{1}{\varepsilon} W_{-1}$. Clearly ξ is of order one. We want this to be true also of η and ζ . We must therefore restrict the values of u_2 and u_3 for which f^- is non-zero to lie almost entirely within some intervals of V_{-1} and W_{-1} . These intervals may be large but, strictly speaking, must be independent of ε . That is, the electron temperature must be uniformly bounded.

In the new variables (2.19) takes the form

$$\begin{aligned}
(2.23) \quad & \frac{\partial \hat{f}}{\partial t} + (\xi + U_0) \frac{\partial \hat{f}}{\partial r} + \left[\frac{(\eta + \frac{1}{\varepsilon} V_{-1})^2}{r} - \frac{1}{\varepsilon} (E_1 + B_3) (\eta + \frac{1}{\varepsilon} V_{-1}) \right. \\
& \left. - B_2 (\zeta + \frac{1}{\varepsilon} W_{-1}) \right] \hat{f}_\xi - \frac{1}{\varepsilon} [E_2 - (\xi + U_0) B_3 + \varepsilon \frac{1}{r} (\xi + U_0) (\eta + \frac{1}{\varepsilon} V_{-1})] \hat{f}_\eta \\
& - \frac{1}{\varepsilon} [E_3 - (\xi + U_0) B_2] \hat{f}_\zeta = 0 .
\end{aligned}$$

where $\hat{f}(t, r, \xi, \eta, \zeta) = f^-(t, r, u_1, u_2, u_3)$ and $\partial \hat{f} / \partial t = \hat{f}_t + \xi_t \hat{f}_\xi + \eta_t \hat{f}_\eta + \zeta_t \hat{f}_\zeta$ with a similar formula for $\partial \hat{f} / \partial r$. Using (2.22)

this can be written

$$\begin{aligned}
 & \frac{\partial \hat{f}}{\partial t} + (\xi + U_0) \frac{\partial \hat{f}}{\partial r} + \left[\frac{1}{\varepsilon} \left(-\frac{2V_{-1}\eta}{r} - B_3\eta + B_2\xi \right) + \frac{\eta^2}{r} \right] \hat{f}_\xi \\
 (2.24) \quad & + \left[\frac{1}{\varepsilon} (V_{-1t} + U_0 V_{-1r} + \xi B_3 - \xi \frac{V_{-1}}{r}) + \frac{\eta}{r} (\xi + U_0) \right] \hat{f}_\eta \\
 & + \left[\frac{1}{\varepsilon} (W_{-1t} + U_0 W_{-1r} - \xi B_2) \right] \hat{f}_\zeta = 0 .
 \end{aligned}$$

In this translated velocity space we define elliptical coordinates Ω , θ , and ϕ by

$$\Omega^2 = \frac{\xi^2}{P^2} + \frac{\eta^2}{Q^2} + \frac{\zeta^2}{S^2} , \quad \tan \theta = \frac{P\eta}{Q\xi} , \quad \tan^2 \phi = \frac{(SQ\xi)^2 + (SP\eta)^2}{(PQ\xi)^2}$$

where P , Q and S are as yet undetermined functions of r and t . Ω is a measure of the temperature of the electrons. These may be inverted to yield

$$\xi = P\Omega \cos \theta \sin \phi, \quad \eta = Q\Omega \sin \theta \sin \phi, \quad \zeta = S\Omega \cos \phi.$$

Letting $F(t, r, \Omega, \theta, \phi) = \hat{f}(t, r, \xi, \eta, \zeta)$ we have the following identities:

$$\hat{f}_\xi = \frac{\cos \theta \sin \phi}{P} F_\Omega - \frac{\sin \theta}{\Omega P \sin \phi} F_\theta + \frac{\cos \theta \cos \phi}{\Omega P} F_\phi$$

$$\hat{f}_\eta = \frac{\sin \theta \sin \phi}{Q} F_\Omega + \frac{\cos \theta}{\Omega Q \sin \phi} F_\theta + \frac{\sin \theta \cos \phi}{\Omega Q} F_\phi$$

$$\hat{f}_\zeta = \frac{\cos \phi}{S} F_\Omega - \frac{\sin \phi}{\Omega S} F_\phi$$

$$\begin{aligned}
 \frac{\partial \hat{f}}{\partial t} = & F_t - F_\Omega \left[\Omega \frac{P_t}{P} \cos^2 \theta \sin^2 \phi + \Omega \frac{Q_t}{Q} \sin^2 \theta \sin^2 \phi + \Omega \frac{S_t}{S} \cos^2 \phi \right. \\
 & \left. - \frac{\xi_t}{P} \cos \theta \sin \phi - \frac{\eta_t}{Q} \sin \theta \sin \phi - \frac{\zeta_t}{S} \cos \phi \right] \\
 & + F_\theta \left[\left(\frac{P_t}{P} - \frac{Q_t}{Q} \right) \sin \theta \cos \theta - \frac{\xi_t \sin \theta}{P\Omega \sin \phi} + \frac{\eta_t \cos \theta}{Q\Omega \sin \phi} \right] +
 \end{aligned}$$

$$+ F_{\phi} \left[\frac{1}{PQS\Omega} \{ (\cos \theta (SQ\xi)_t + \sin \theta (SP\eta)_t) \cos \phi - (PQ\xi)_t \sin \phi \} \right] .$$

A similar formula holds for $\partial \hat{f} / \partial r$.

When these differential identities are substituted into (2.24) we find that by setting

$$(2.25) \quad \frac{P^2}{Q^2} = \frac{rB_3 - 2V_{-1}}{rB_3 - (rV_{-1})_r} , \quad \frac{P^2}{S^2} = \frac{B_2}{B_2 + (W_{-1})_r} ,$$

the coefficient of $\frac{1}{\varepsilon} F_{\Omega}$ vanishes. Equation (2.24) becomes

$$(2.26) \quad F_t + (P_{\Omega} \cos \theta \sin \phi + U_0) F_r + \mathcal{D} F_{\Omega} + \left(\frac{1}{\varepsilon} \mathcal{E}' + \mathcal{D}' \right) F_{\theta} + \left(\frac{1}{\varepsilon} \mathcal{E}'' + \mathcal{D}'' \right) F_{\phi} = 0$$

where

$$\begin{aligned} \mathcal{D} &= -\Omega^2 P_r \cos^3 \theta \sin^3 \phi + \Omega^2 P \left(\frac{1}{r} \frac{Q^2}{P^2} - \frac{Q_r}{Q} - \frac{1}{r} \right) \sin^2 \theta \cos \theta \sin^3 \phi \\ &\quad - \Omega \left(\frac{P_t}{P} + U_0 \frac{P_r}{P} + U_{0r} \right) \cos^2 \theta \sin^2 \phi - \Omega \left(\frac{Q_t}{Q} + U_0 \frac{Q_r}{Q} + \frac{U_0}{r} \right) \\ &\quad \cdot \sin^2 \theta \sin^2 \phi - \Omega \left(\frac{S_t}{S} + U_0 \frac{S_r}{S} \right) \cos^2 \phi - \Omega^2 \frac{PS_r}{S} \cos \theta \sin \phi \cos^2 \phi \\ &\quad - \frac{1}{P} (U_{0t} + U_0 U_{0r}) \cos \theta \sin \phi \\ \mathcal{E}' &= \frac{P}{Q} (rB_3 - (rV_{-1})_r) - \frac{P}{S} (B_2 + W_{-1r}) \frac{\sin \theta \cos \phi}{\sin \phi} \\ \mathcal{D}' &= \frac{1}{\Omega P} (U_{0t} + U_0 U_{0r}) \frac{\sin \theta}{\sin \phi} + \left(\frac{P_t}{P} - \frac{Q_t}{Q} + U_0 \left(\frac{P_r}{P} - \frac{Q_r}{Q} \right) \right. \\ &\quad \left. - \frac{U_0}{r} + U_{0r} \right) \sin \theta \cos \theta + \Omega P \left(\frac{P_r}{P} - \frac{Q_r}{Q} - \frac{1}{r} \right) \sin \theta \cos^2 \theta \sin \phi \\ &\quad - \frac{\Omega Q^2}{rP} \sin^3 \theta \sin \phi \\ \mathcal{E}'' &= \frac{P}{S} (B_2 + W_{-1r}) \end{aligned}$$

$$\begin{aligned}
\mathcal{D}'' = & \left(\frac{S_t}{S} - \frac{P_t}{P} + U_0 \left(\frac{S_r}{S} - \frac{P_r}{P} \right) \right) \cos \phi \sin \phi \\
& + \Omega P \left(\frac{S_r}{S} + \frac{Q_r}{Q} \right) \cos^3 \theta \sin^3 \phi \cos \phi \\
& - \Omega S \left(\frac{P_r}{P} + \frac{Q_r}{Q} \right) \cos \theta \sin^2 \phi \cos \phi \\
& + \Omega P \left(\frac{Q P_r}{P^2} + \frac{Q S_r}{P S} + \frac{Q^2}{P^2 r} - \frac{1}{r} \right) \sin^2 \theta \cos \theta \sin^2 \phi \cos \phi \\
& - \frac{1}{\Omega P} (U_{0_t} + U_0 U_{0_r}) \cos \theta \cos \phi - U_{0_r} \cos^2 \theta \cos \phi \sin \phi \\
& - \frac{U_0}{r} \sin^2 \theta \cos \phi \sin \phi.
\end{aligned}$$

Equation (2.26) appears to have several singularities occurring at $r = 0$. However, if we take the natural conditions that V^- and U vanish at the origin we find that the term $\frac{1}{r} \left(\frac{Q^2}{P^2} - 1 \right)$ which appears in $\mathcal{D}$ and $\mathcal{D}'$ can be written as

$$\frac{1}{r} \left(\frac{Q^2}{P^2} - 1 \right) = \frac{1}{r} \left(\frac{r B_{\mathcal{Z}} - 2 V_{-1}}{r B_{\mathcal{Z}} - (r V_{-1})_r} - 1 \right) = \frac{\left(\frac{1}{r} V_{-1} \right)_r}{B_{\mathcal{Z}} + \frac{1}{r} (r V_{-1})_r}.$$

This will be regular if P/Q is. We shall later solve for P , Q and S explicitly and verify that this quotient is well behaved. Since U_0/r is regular the only true singularity appears in $\mathcal{D}'$ giving the condition that $F_\theta(t, 0, \Omega, \theta, \phi) = 0$.

We seek now a series representation of F in the form

$$F = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \dots.$$

This is substituted into (2.26) and the coefficient of each power of ε set equal to zero. The first equation yielded is

$$\begin{aligned}
& + \frac{1}{P} (U_{0_t} + U_0 U_{0_r}) \sin \theta \sin \phi] \mathcal{F}_{0_\Omega} \\
= & - \Gamma' F_1 - \Gamma'' \left(\frac{\partial}{\partial \phi} \int_0^\theta \cos \theta F_1 d\theta + \frac{\cos \phi}{\sin \phi} \int_0^\theta \cos \theta F_1 d\theta \right. \\
& \left. - \frac{\cos \phi}{\sin \phi} \sin \theta F_1 \right) + \mathcal{F}_1(t, r, \Omega, \phi) .
\end{aligned}$$

We make two definitions:

$$L(t, r, \Omega, \phi) = \int_0^{2\pi} \cos \theta F_1 d\theta, \quad L'(t, r, \Omega, \phi) = \int_0^{2\pi} \cos \theta \sin \theta F_1 d\theta.$$

Using the fact that F_1 must be periodic in θ we evaluate

(2.29) at $\theta = 2n\pi$, where n is an integer to obtain

$$\begin{aligned}
(2.30) \quad & 2n\pi(\mathcal{F}_{0_t} + U_0 \mathcal{F}_{0_r}) - \Omega n\pi \left[\left(\frac{P_t}{P} + \frac{Q_t}{Q} \right) + U_0 \left(\frac{P_r}{P} + \frac{Q_r}{Q} \right) \right. \\
& \left. + U_{0_r} + \frac{U_0}{r} \right] \sin^2 \phi + 2 \left(\frac{S_t}{S} + U_0 \frac{S_r}{S} \right) \cos^2 \phi] \mathcal{F}_{0_\Omega} \\
& = - \Gamma' F_1(t, r, \Omega, 2n\pi, \phi) - n\Gamma'' (L_\phi + \frac{\cos \phi}{\sin \phi} L) + \mathcal{F}'_1(t, r, \phi).
\end{aligned}$$

Setting $n = 0$ gives $\mathcal{F}'_1(t, r, \Omega, \phi) = F_1(t, r, \Omega, 0, \phi)$. Then

for n nonzero we obtain from equation (2.30)

$$\begin{aligned}
(2.31) \quad & 2(\mathcal{F}_{0_t} + U_0 \mathcal{F}_{0_r}) - \Omega \left[\left(\frac{P_t}{P} + \frac{Q_t}{Q} \right) + U_0 \left(\frac{P_r}{P} + \frac{Q_r}{Q} \right) + U_{0_r} + \frac{U_0}{r} \right] \sin^2 \phi \\
& + 2 \left(\frac{S_t}{S} + U_0 \frac{S_r}{S} \right) \cos^2 \phi] \mathcal{F}_{0_\Omega} = - \frac{\Gamma''}{\pi \sin \phi} \frac{\partial}{\partial \phi} (L \sin \phi).
\end{aligned}$$

Multiplying this by $\sin \phi$ and integrating with respect to ϕ from $\phi = 0$ gives

$$\begin{aligned}
\sin \phi L &= \frac{2\pi}{\Gamma'} \cos \phi (\mathcal{F}_{0_t} + U_0 \mathcal{F}_{0_r}) \\
&+ \frac{\Omega\pi}{\Gamma''} \left[\left(\frac{P_t}{P^-} + \frac{Q_t}{Q^-} + U_0 \left(\frac{P_r}{P^-} + \frac{Q_r}{Q^-} \right) + U_{0_r} + \frac{U_0}{r} \right) \left(\frac{\cos^3 \phi}{3} - \cos \phi \right) \right. \\
(2.32) \quad &- \left. 2 \left(\frac{S_t}{S^-} + U_0 \frac{S_r}{S^-} \right) \frac{\cos^3 \phi}{3} \right] \mathcal{F}_{0_\Omega} - g(t, r, \Omega) \quad \text{where} \\
g &= \frac{2\pi}{\Gamma'} (\mathcal{F}_{0_t} + U_0 \mathcal{F}_{0_r}) - \frac{2\pi}{3\Gamma''} \left[\frac{P_t}{P^-} + \frac{Q_t}{Q^-} + \frac{S_t}{S^-} \right. \\
&+ \left. U_0 \left(\frac{P_r}{P^-} + \frac{Q_r}{Q^-} + \frac{S_r}{S^-} \right) + U_{0_r} + \frac{U_0}{r} \right] \mathcal{F}_{0_\Omega} .
\end{aligned}$$

Evaluating (2.32) at $\phi = \pi$ gives simply $2g = 0$, so that, since g is independent of ϕ , it is identically zero. We still have one relationship in P , Q and S at our disposal. By taking

$$(2.33) \quad PQS = A(Y)N, \text{ with } A \text{ an undetermined function of } Y,$$

the second term in g vanishes leaving

$$(2.34) \quad \mathcal{F}_{0_t} + U_0 \mathcal{F}_{0_r} = 0 .$$

Equation (2.32) with g set equal to zero gives an explicit expression for L . This will be needed to determine moments of the distribution function. We shall also need an expression for L' . For this we multiply (2.28) by $\sin \theta$ and integrate with respect to θ from 0 to 2π , taking (2.32) into account. After integrating by parts we obtain

$$(2.35) \quad \Gamma''(\sin^2 \phi L')_\phi = \Gamma' \sin \phi L .$$

Upon solving for L and L' we obtain

$$(2.36) \quad L = \frac{\Omega\pi}{\Gamma''} \left(\frac{S_t}{S} + U_0 \frac{S_r}{S} \right) \mathcal{F}_{0\Omega} \sin \phi \cos \phi$$

$$L' = \frac{\Omega\pi\Gamma'}{3\Gamma''^2} \left(\frac{S_t}{S} + U_0 \frac{S_r}{S} \right) \mathcal{F}_{0\Omega} \sin \phi .$$

From equations (2.25) and (2.33) we find

$$(2.37) \quad \begin{aligned} P^6 &= (AN)^2 \frac{rB_3 - 2V_{-1}}{rB_3 - (rV_{-1})_r} \frac{B_2}{B_2 + W_{-1}_r} \\ Q^6 &= (AN)^2 \left(\frac{rB_3 - (rV_{-1})_r}{rB_3 - 2V_{-1}} \right)^{-2} \frac{B_2}{B_2 + W_{-1}_r} \\ S^6 &= (AN)^2 \frac{rB_3 - (rV_{-1})_r}{rB_3 - 2V_{-1}} \left(\frac{B_2}{B_2 + W_{-1}_r} \right)^{-2} . \end{aligned}$$

We are now able to evaluate, to lowest order in ε , the moments of the electron distribution function. For the mean radial velocity we find

$$U \equiv \frac{\int u_1 f^- |d\bar{u}|}{\int f^- |d\bar{u}|} = \frac{\int_0^\infty \int_0^{2\pi} \int_0^\pi (P\Omega \cos \theta \sin \phi + U_0) \Omega^2 PQS \sin^2 \phi F \, d\phi \, d\Omega \, d\theta}{\int_0^\infty \int_0^{2\pi} \int_0^\pi \Omega^2 PQS \sin^2 \phi F \, d\phi \, d\theta \, d\Omega}$$

$$= U_0 + \frac{1}{N} \int_0^\infty \int_0^{2\pi} \int_0^\pi \Omega^3 \cos \theta \sin^3 \phi (\mathcal{F}'_0 + \varepsilon F_1) \, d\phi \, d\theta \, d\Omega + O(\varepsilon^2)$$

$$= U_0 + \frac{\mathcal{F}_0}{N} \int_0^\infty \int_0^{2\pi} \int_0^\pi \Omega^3 \cos \theta \sin^3 \phi \, d\phi \, d\theta \, d\Omega$$

$$+ \frac{\varepsilon}{N} \int_0^\infty \int_0^\pi \Omega^3 \sin^3 \phi L \, d\phi \, d\Omega + O(\varepsilon^2) .$$

The first integral can be evaluated directly and is seen to vanish. Using the expression for L in (2.36) the second integral is seen also to vanish. The remaining velocities are found in a similar manner yielding

$$(2.38) \quad U = U_0 + O(\varepsilon^2), \quad V^- = \frac{1}{\varepsilon} V_{-1} + O(\varepsilon), \quad W^- = \frac{1}{\varepsilon} W_{-1} + O(\varepsilon).$$

Using (2.38) the electron pressure terms needed in the equations of motion are evaluated:

$$\begin{aligned} P_{11}^- &= \frac{\varepsilon^2}{1+\varepsilon^2} \int (u_1 - U)^2 f^- |d\bar{u}| \\ &= \frac{\varepsilon^2}{1+\varepsilon^2} [P^3_{QS} \int_0^\infty \int_0^{2\pi} \int_0^\pi \Omega^4 \cos^2 \theta \sin^4 \phi F d\phi d\theta d\Omega + O(\varepsilon^2)] \end{aligned}$$

$$\begin{aligned} P_{22}^- &= \frac{\varepsilon^2}{1+\varepsilon^2} \int (u_2 - V^-)^2 f^- |d\bar{u}| \\ &= \frac{\varepsilon^2}{1+\varepsilon^2} [PQ^3_S \int_0^\infty \int_0^{2\pi} \int_0^\pi \Omega^4 \sin^2 \theta \sin^4 \phi F d\phi d\theta d\Omega + O(\varepsilon)] \end{aligned}$$

$$\begin{aligned} P_{12}^- &= \frac{\varepsilon^2}{1+\varepsilon^2} \int (u_1 - U)(u_2 - V^-) f^- |d\bar{u}| \\ &= \frac{\varepsilon^2}{1+\varepsilon^2} [P^2_Q{}^2_S \int_0^\infty \int_0^{2\pi} \int_0^\pi \Omega^4 \cos \theta \sin \theta \sin^4 \phi F d\phi d\theta d\Omega + O(\varepsilon^2)] \end{aligned}$$

$$\begin{aligned} P_{13}^- &= \frac{\varepsilon^2}{1+\varepsilon^2} \int (u_1 - U)(u_3 - W^-) f^- |d\bar{u}| \\ &= \frac{\varepsilon^2}{1+\varepsilon^2} [P^2_Q{}^2_S \int_0^\infty \int_0^{2\pi} \int_0^\pi \Omega^4 \cos \theta \cos \phi \sin^3 \phi F d\phi d\theta d\Omega + O(\varepsilon^2)] \end{aligned}$$

When the lowest order expression for F is substituted into these expressions we can integrate to obtain

$$\begin{aligned}
P_{11}^- &= \varepsilon^2 \frac{3\pi^2}{8} P_{QS}^3 \int_0^\infty \Omega^4 \mathcal{F}_0 d\Omega + o(\varepsilon^3) \\
P_{22}^- &= \varepsilon^2 \frac{3\pi^2}{8} P_{QS}^3 \int_0^\infty \Omega^4 \mathcal{F}_0 d\Omega + o(\varepsilon^3) \\
P_{12}^- &= \varepsilon^3 P_Q^2 Q^2 S \int_0^\infty \int_0^\pi \Omega^4 L' \sin^4 \phi d\phi d\Omega + o(\varepsilon^4) \\
(2.39) \quad &= - \varepsilon^3 \frac{16\pi\Gamma'}{9\Gamma''^2} (S_t + US_r) P_Q^2 Q^2 \int_0^\infty \Omega^4 \mathcal{F}_0 d\Omega + o(\varepsilon^4) \\
P_{13}^- &= \varepsilon^3 P_Q^2 Q S^2 \int_0^\infty \int_0^\pi \Omega^4 L \cos \phi \sin^3 \phi d\phi d\Omega + o(\varepsilon^4) \\
&= - \varepsilon^3 \frac{5\pi^2}{16\Gamma''} (S_t + US_r) P_Q^2 Q S \int_0^\infty \Omega^4 \mathcal{F}_0 d\Omega + o(\varepsilon^4)
\end{aligned}$$

where we have used (2.36) and (2.38).

It is necessary here to discuss the relative magnitudes of the ion and electron pressure terms. The ε^2 multiplying the electron terms would seem to indicate that these terms are small compared to the ion terms and that since we have already neglected terms of order ε we should neglect these as well. This is, however, not realistic. The average kinetic energy per particle must be the same for the electrons and ions. In a collisional theory this would be true and although our model is collisionless it can be thought of as arising by allowing a collision parameter to go to zero. To be consistent with our asymptotic development and yet retain both pressures in the equations it is necessary to assume that the quantity $\varepsilon^2 \int_0^\infty \Omega^4 \mathcal{F}_0 d\Omega$ is, although strictly speaking still $o(\varepsilon^2)$, sufficiently large

to allow the electron pressure terms to be comparable with the ion terms. Since $\int_0^\infty \Omega^2 \mathcal{F}_0 d\Omega$ is the number density, and thus approximately one, we must assume that the thermal speed of the electrons is high. We define the quantity α^- to be

$$(2.40) \quad \alpha^-(Y) = \varepsilon^2 \frac{3\pi^2}{8} \int \Omega^4 \mathcal{F}_0 d\Omega ,$$

and assume that α^- is sufficiently large that it cannot be neglected compared to one. With this assumption the electron pressure terms are of the same magnitude as the ion terms. Equation (2.34) shows that α^- is indeed independent of T . We then have

$$(2.41) \quad \begin{aligned} P_{11}^- &= \alpha^- P^3 Q S + O(\varepsilon^3) \\ P_{22}^- &= \alpha^- P Q^3 S + O(\varepsilon^3) \\ P_{12}^- &= - \varepsilon \frac{128}{27\pi \Gamma''^2} (S_t + U S_r) \alpha^- P^2 Q^2 + O(\varepsilon^4) \\ P_{13}^- &= - \varepsilon \frac{5}{6\Gamma''} (S_t + U S_r) \alpha^- P^2 Q S + O(\varepsilon^4) . \end{aligned}$$

d. The electron pressure tensor, theta and pure z-pinches

To study the theta pinch in which B_2 , E_3 , W^+ , and W^- are all identically zero we observe that since there is now no force in the z direction, S is identically one. By setting $S = 1$ in (2.37) we see that in the limit where B_2 and W^- tend to zero we have

$$\left(\frac{B_2}{B_2 + W_{-1r}} \right)^2 \rightarrow (AN)^2 \frac{rB_3 - 2V_{-1}}{rB_3 - (rV_{-1})_r} .$$

Using this we obtain from (2.37)

$$(2.42) \quad P^4 = (AN)^2 \frac{rB_3 - 2V_{-1}}{rB_3 - (rV_{-1})_r}$$

$$Q^4 = (AN)^2 \left(\frac{rB_3 - 2V_{-1}}{rB_3 - (rV_{-1})_r} \right)^{-1}.$$

P_{11}^- and P_{22}^- are now easily found from (2.42) and (2.41) with S set equal to one. To find P_{12}^- we must refer back to (2.28) which now becomes

$$(2.43) \quad \mathcal{F}_{0_t} + (P\Omega \cos \theta \sin \phi + U_0) \mathcal{F}_{0_r} + \mathcal{D}\mathcal{F}_{0_\Omega} + \Gamma^1 F_{1_\theta} = 0.$$

Using the fact that $(PQ/N)_T = 0$ we integrate (2.43) with respect to θ and find F_1 . From this the pressure terms appropriate for the theta pinch are found to be

$$(2.44) \quad P_{11}^- = \alpha^- P^3 Q + O(\epsilon^3)$$

$$P_{22}^- = \alpha^- P Q^3 + O(\epsilon^3)$$

$$P_{12}^- = \epsilon \frac{24}{25\pi} \frac{P^2 Q^2}{\Gamma^1} - \alpha^- \left(\frac{P_t}{P} + U \frac{P_r}{P} + U_r \right) + O(\epsilon^4).$$

P_{13}^- is no longer needed in the equations of motion.

When B_3 , E_2 , V^+ , and V^- are all identically zero we have a pure z pinch. By setting Q equal to one we find that in the limit when B_3 goes to zero

$$\left(\frac{rB_3 - 2V_{-1}}{rB_3 - (rV_{-1})_r} \right)^2 \longrightarrow (AN)^2 \frac{B_2}{B_2 + W_{-1}_r}.$$

For this case we obtain from (2.37)

$$\begin{aligned}
 (2.45) \quad P^4 &= (AN)^2 \frac{B_2}{B_2 + W_{-1} r} \\
 S^4 &= (AN)^2 \left(\frac{B_2}{B_2 + W_{-1} r} \right)^{-1}
 \end{aligned}$$

As before, we find P_{11}^- and P_{22}^- from (2.45) and (2.41) with now Q set equal to one. P_{13}^- requires no special care in this limit. The electron pressure terms for the pure z pinch are given by

$$\begin{aligned}
 (2.46) \quad P_{11}^- &= \alpha^- P^3 S + O(\epsilon^3) \\
 P_{22}^- &= \alpha^- P S + O(\epsilon^3) \\
 P_{13}^- &= -\epsilon \frac{5}{6} \frac{P^2 S}{F\pi} \alpha^- (S_t + U S_r) + O(\epsilon^4)
 \end{aligned}$$

For the pure z pinch P_{12}^- is not needed.

e. The systems to be solved

Having found expressions for the ion and electron pressure tensors the equations of motion (2.13) and (2.14) are now closed. To incorporate the electron pressure terms into the equations of motion we must first express them in Lagrangian coordinates. We then find

$$\begin{aligned}
 H_2 &= R_Y B_2 - (R B_2)_{YY} \\
 H_3 &= R R_Y B_3 - (R^2 B_3)_Y \\
 (2.47) \quad P^6 &= (AN)^2 \left(1 + \frac{(R B_2)_{YY}}{H_2} \right) \left(1 + \frac{R^2 B_3_{YY}}{H_3} \right) + O(\epsilon)
 \end{aligned}$$

$$S^6 = (AN)^2 \left(1 + \frac{(RB_2)_{YY}}{H_2}\right)^{-2} \left(1 + \frac{R^2 B_3 YY}{H_3}\right) + O(\varepsilon)$$

$$Q^6 = (AN)^2 \left(1 + \frac{(RB_2)_{YY}}{H_2}\right) \left(1 + \frac{R^2 B_3 YY}{H_3}\right)^{-2} + O(\varepsilon)$$

$$\Gamma' = P^2 S H_3 = O(\varepsilon)$$

$$\Gamma'' = \frac{PH_2}{SR_Y} + O(\varepsilon)$$

$$A_T = 0$$

When these terms and the expressions for the ion pressure are substituted into (2.13) and (2.14) we obtain, after eliminating Q, the following closed system of equations.

$$(2.48) \quad \begin{aligned} & R_{TT} - R(B_3 Y)^2 + B_2 (RB_2)_Y + R B_3 B_3 Y \\ & + \left(\frac{\alpha^+}{2YR_Y}\right)_Y - \frac{2\alpha^+ Y}{R^3} + \left(\frac{\alpha^- AP^2}{R_Y}\right)_Y - \frac{\alpha^- AQ^2}{R} = O(\varepsilon) \\ & H_{2T} - \frac{5}{12} \left(\frac{\alpha^- AS_T^2}{H_2}\right)_{YY} = O(\varepsilon) \\ & H_{3T} + \frac{64}{27\pi} \left(\frac{\alpha^- A^2 H_3 S_T^2}{H_2^2}\right)_{YY} = O(\varepsilon) \end{aligned}$$

For the theta pinch in which only the B_3 field is present we obtain

$$(2.49) \quad \begin{aligned} P^4 &= (AN)^2 \left(1 + \frac{R^2 B_3 YY}{H_3}\right) + O(\varepsilon) \\ Q^4 &= (AN)^2 \left(1 + \frac{R^2 B_3 YY}{H_3}\right)^{-1} + O(\varepsilon) \\ \Gamma' &= P^3 H_3 + O(\varepsilon) \\ A_T &= 0 \end{aligned}$$

The equations of motion become

$$\begin{aligned}
 (2.50) \quad R_{TT} - R(B_3)_Y^2 + R B_3 B_{3Y} + \left(\frac{\alpha^+}{2Y R_Y}\right)_Y - \frac{2\alpha^+ Y}{R^3} \\
 + \left(\frac{\alpha^- A P^2}{R_Y}\right)_Y - \frac{\alpha^- A Q^2}{R} = O(\varepsilon) \\
 H_{3T} + \frac{25}{48\pi} \left(\frac{\alpha^- A^2}{H_3} \left(\frac{1}{P^2 R_Y^2}\right)_T\right)_{YY} = O(\varepsilon)
 \end{aligned}$$

And finally, the pure z pinch expressions become

$$\begin{aligned}
 (2.51) \quad P^4 &= (AN)^2 \left(1 + \frac{(RB_2)_{YY}}{H_2}\right) + O(\varepsilon) \\
 S^4 &= (AN)^2 \left(1 + \frac{(RB_2)_{YY}}{H_2}\right)^{-1} + O(\varepsilon) \\
 \Gamma'' &= \frac{PH_2}{SR_Y} + O(\varepsilon) \\
 A_T &= 0,
 \end{aligned}$$

and the corresponding equations of motion are

$$\begin{aligned}
 (2.52) \quad R_{TT} + B_2 (RB_2)_Y + \left(\frac{\alpha^+}{2Y R_Y}\right)_Y - \frac{2\alpha^+ Y}{R^3} \\
 + \left(\frac{\alpha^- A P^2}{R_Y}\right)_Y - \frac{\alpha^- A}{R} = O(\varepsilon) \\
 H_{2T} - \frac{5}{12} \left(\frac{\alpha^- A S_T^2}{H_2}\right)_{YY} = O(\varepsilon).
 \end{aligned}$$

Systems (2.48), (2.50) and (2.52) are solved with the following initial and boundary conditions. We assume that initially there is an infinite cylinder of plasma at equilibrium in which the density is everywhere equal to one and there are no currents in the plasma. Thus, B_2 is

identically zero while B_3 is constant. To initiate the motion an increasing magnetic field is applied at the outer boundary of the plasma causing the radial contraction.

When pressure is added to the problem, however, we find that we have more free variables than we need, for we now have $\alpha(Y)$, and $A(Y)$ which are as yet undetermined. If we assume that the initial electron distribution function is symmetric about the mean velocity we have by definition that P , Q and S are constant. From (2.33) we then find that $A(Y)$ is also constant. Without loss of generality we take these constants to be one. When the desired initial values of R , B_2 and B_3 are substituted into the equilibrium equation obtained from setting all T derivatives equal to zero in (2.48) we find that

$$2Y(\alpha^+ + \alpha^-)_Y = 0 .$$

Since these terms are positive, they must be constant.

The initial and boundary conditions for the z and theta pinches associated with systems (2.48), (2.50), and (2.52) are discussed in Appendix 3. They are

$$\begin{aligned}
 N &\equiv \frac{1}{RR_Y} = 1 \Rightarrow R(0,Y) = \sqrt{2Y} \\
 R_T(-0,Y) &= B_{2T}(-0,Y) = B_{3T}(-0,Y) = 0 \\
 B_2(0,Y) &= 0 \\
 B_3(0,Y) &= \begin{cases} 1 & \text{for (2.48) and (2.50)} \\ 0 & \text{for (2.52)} \end{cases} \\
 (2.53) \quad R(T,0) &= 0
 \end{aligned}$$

$$B_2(T,0) = 0$$

$$B_3(T,0) \text{ regular}$$

$$B_2(T, Y_M) \text{ prescribed ; } (RP_{11}^+)_Y - R_Y P_{22}^+ \Big|_{Y=Y_M} = 0$$

$$B_3(T, Y_M) \text{ or } B_{3Y}(T, Y_M) \text{ prescribed.}$$

Y_m is the value of Y at the outer boundary of the plasma.

We shall consider also some pressure-free flows in which both fields are initially present in a force-free configuration. Let $J_3(0, Y)$ be a constant denoted by $\frac{1}{2} m^2$. With this definition $B_2(0, Y) = mY$ and (2.53) must be altered by letting

$$B_2(0, Y) = mY \text{ and}$$

$$(2.53a) \quad B_3(0, Y) \text{ satisfy } (B_{3Y})^2 - B_3 B_{3Y} - m^2 = 0$$

$$\text{with } B_3^2(0, Y_m) + B_2^2(0, Y_m) = 1 .$$

The quantities H_2 and H_3 defined in (2.47) are seen to arise in two different contexts, indicating that they have significance in their own right. This notion is enforced by the fact that when the electron pressure is absent they are both invariant in time. Their relationship to the more familiar one-fluid fields can be seen by writing them as

$$H_2 = \frac{B_2}{RN} - (R B_2)_{YY} \text{ and } H_3 = \frac{B_3}{N^2} - (R B_{3Y})_Y .$$

Thus, when the differentiated terms on the right are small, or when they vanish as they do initially, the near invariance

of these terms says that $B_2 \propto RN$ and $B_3 \propto N$. These are the expressions for the fields in the one-fluid model. From (2.48) we see also that the two field equations are coupled by these quantities.

f. Characteristics

It is clear that the inclusion of electron pressure greatly complicates the equations of motion. It also raises some new questions. For instance, since the highest derivatives now appear with a factor of α^- do the equations become singular as this parameter goes to zero? Furthermore, from the definitions of P given in (2.47), (2.49), and (2.51) for the different variations of the problem we see that while in (2.48) and (2.52) the first equation contains R_{TT} and R_{YY} as the highest derivatives, the corresponding equation in (2.50) contains only R_{TT} and R_{YY} . Does this mean that the two systems are of different type? Finally, are the three systems with initial and boundary conditions as given in (2.53) and (2.53a) well posed? To answer these questions we must examine the characteristics of the three systems.

The characteristics of the system (2.48) and the special cases derived from it are determined in Appendix 1. The results are given here. Let

$$C_L^{+2} = \frac{3\alpha^+}{2YR_Y^4} \quad \text{and} \quad C_L^{-2} = \frac{3\alpha^-P^2}{R_Y^2} \quad .$$

These are the sound speeds corresponding to the two pressures. They are given in the Lagrangian representation, dY/dT . To obtain the more familiar Eulerian form it is necessary to multiply each by R_Y . The characteristics for the three systems with and without pressure are given by:

Pure z pinch-system (2.52)

$$\left. \begin{array}{l} \alpha^+ = 0 \\ \alpha^- = 0 \end{array} \right\} \quad dT^2 dY^3 = 0$$

$$\left. \begin{array}{l} \alpha^+ > 0 \\ \alpha^- = 0 \end{array} \right\} \quad dT^2 dY [dY^2 - c_L^{+2} dT^2] = 0$$

$$\left. \begin{array}{l} \alpha^+ > 0 \\ \alpha^- > 0 \end{array} \right\} \quad dT^4 dY [dY^2 - (c_L^{+2} + c_L^{-2}) dT^2] = 0$$

Theta pinch - system (2.50)

$$\left. \begin{array}{l} \alpha^+ = 0 \\ \alpha^- = 0 \end{array} \right\} \quad dT^2 dY^3 = 0$$

$$\left. \begin{array}{l} \alpha^+ > 0 \\ \alpha^- = 0 \end{array} \right\} \quad dT^2 dY [dY^2 - c_L^{+2} dT^2] = 0$$

$$\left. \begin{array}{l} \alpha^+ > 0 \\ \alpha^- > 0 \end{array} \right\} \quad dT^4 dY [dY^2 - (c_L^{+2} + c_L^{-2}) dT^2] = 0$$

Both fields - system (2.48)

$$\left. \begin{array}{l} \alpha^+ = 0 \\ \alpha^- = 0 \end{array} \right\} \quad dT^4 dY^4 = 0$$

$$\left. \begin{array}{l} \alpha^+ > 0 \\ \alpha^- = 0 \end{array} \right\} \quad dT^4 dY^2 [dY^2 - c_L^{+2} dT^2] = 0$$

$$\left. \begin{array}{l} \alpha^+ > 0 \\ \alpha^- > 0 \end{array} \right\} \quad dT^6 dY^2 [dY^2 - (c_L^{+2} + c_L^{-2}) dT^2] = 0$$

By the nature of the characteristics we see that the problems are essentially fluid-dynamical in nature, the only non-trivial characteristics being the sound speeds. The electron sound speed, like the electron pressure, depends on the magnetic fields as well as on the density. Notice that the Alfven speed does not occur as a characteristic as it does in the one-fluid magnetohydrodynamic model. We see also that the characteristic equations are well behaved when the electron pressure goes to zero. The vanishing of α^- causes only the disappearance of the electron sound speed from the characteristic equations. Also, as was to be expected, there is no basic difference in the characteristics between the theta pinch and the other two problems. The third derivatives in R which appear in the momentum equation of these systems is eliminated by virtue of the fact that the highest order parts of the field equations are not independent of these terms. Finally, since the systems have only these fluid-dynamical characteristics, the systems (2.48), (2.50) and (2.52) are well posed with the initial and boundary conditions given in (2.53) and (2.53a).

g. An energy integral

When the electron pressure is absent and α^+ is taken to be constant an energy integral is associated with the system (2.48). By combining the equations in the following

manner

$$\begin{aligned}
& 2R_T \{ R_{TT} - R(B_{ZY})^2 + RB_Z B_{ZY} + B_2(RB_2)_Y \\
& + \alpha^+ (\frac{1}{2YR_Y^2})_Y - \frac{2\alpha^+_Y}{R^2} \} - 2B_Z \{ (R^2 B_{ZY})_Y - RR_Y B_Z \}_T \\
& - 2RB_2 \{ (RB_2)_{YY} - R_Y B_2 \}_T = 0
\end{aligned}$$

we arrive at

$$\begin{aligned}
& \{ R_T^2 + RR_Y (B_2^2 + B_Z^2) + R^2 B_{ZY}^2 + (RB_2)_Y^2 \\
(2.54) \quad & + \frac{\alpha^+}{2YR_Y^2} + \frac{2\alpha^+_Y}{R^2} \}_T + \{ RR_T (B_2^2 + B_Z^2) - 2B_Z (R^2 B_{ZY})_T \\
& - 2RB_2 (RB_2)_{YT} + \alpha^+ \frac{R_T}{YR_Y^2} \}_Y = 0 .
\end{aligned}$$

This integral provides a useful check on the accuracy of the numerical calculations and serves as a guide in the proof of the stability of the finite difference scheme.

h. The one-fluid model

In this section we compare our two-fluid model of the pinch with the simpler one-fluid model. To do this we write the Lundquist equations in axially symmetric cylindrical Lagrangian coordinates. Let N be the total fluid number density, V and W the circumferential and axial velocities. The pressure is taken to be proportional to the density squared. Keeping the same scaling as in (2.4) the equations of motion are

$$\begin{aligned}
(2.55) \quad & R_{TT} - \frac{V^2}{R} + B_2(RB_2)_Y + RB_3B_{3Y} + R P_Y = 0 \\
& (R_Y B_2)_T = 0 \\
& (RR_Y B_3)_T = 0 \\
& (RV)_T = 0 \\
& W_T = 0 \\
& (P(RR_Y)^2)_T = 0 .
\end{aligned}$$

In comparing the two models we shall take the pressure to be zero in both. In the two-fluid model this requires that the net momentum in the θ and z directions remains identically zero, the value it had initially. To correspond to this we take V and W to be zero. To facilitate comparison we rescale the independent variables in both models by defining

$$X = \delta^2 Y , \quad \tau = \delta T$$

where δ is a positive number. The two fluid equations of motion as given in (2.48) become

$$\begin{aligned}
(2.56) \quad & R_{\tau\tau} + RB_3B_{3X} + B_2(RB_2)_X = \delta^2 R(B_{3X})^2 \\
& (RR_X B_3)_\tau = \delta^2 (R^2 B_{3X})_{X\tau} \\
& (R_X B_2)_\tau = \delta^2 (RB_2)_{XX\tau}
\end{aligned}$$

while the one-fluid equations (2.55) are

$$R_{\tau\tau} + R B_3 B_{3X} + B_2 (R B_2)_X = 0$$

$$(2.57) \quad (R R_X B_3)_\tau = 0$$

$$(R_X B_2)_\tau = 0 .$$

Upon examining these two systems it is clear that as δ approaches zero the two fluid system (2.56) goes over into (2.57), the one-fluid system. The limiting process is, however, singular as δ multiplies the highest derivatives in the two-fluid field equations. Thus, as Morton (15) points out, the two-fluid model does not allow discontinuities in the magnetic fields while the one-fluid does. Both allow jumps in the fluid variables when pressure is included. As δ goes to zero any oscillations present in the two-fluid model become compressed both in space and time so the limit may be thought of as a space-time average of the oscillation.

Continuing the correspondence between the two models we again take the initial condition to be one of equilibrium. In this case, though, we can find simple formulas for the initial fields. Thus

$$(2.58) \quad R^2(Y,0) = 2Y, \quad B_2^2(Y,0) = m^2 Y, \quad B_3^2(Y,0) = b - 2m^2 Y,$$

where m and b are constants. From (2.55) and (2.58) we have

$$B_2 = \frac{1}{R_Y} - \frac{m}{\sqrt{2}} \quad \text{and} \quad B_3 = \frac{\sqrt{b - 2m^2 Y}}{R R_Y} .$$

From these we can find one second order equation for R:

$$(2.59) \quad R_{TT} + \frac{1}{2} R \left[\frac{b-2m^2 Y}{(RR_Y)^2} \right]_Y + \frac{m^2}{2R_Y} \left(\frac{R}{R_Y} \right)_Y + AR \left[\frac{1}{(RR_Y)^2} \right]_Y = 0 .$$

The characteristics of system (2.55) are given by

$$dY^3 \left[\left(\frac{dY}{dT} \right)^2 - \frac{R}{R_Y} (B_2^2 + B_3^2 + 2P) \right] = 0 .$$

The last term in this is the sum of the squares of the Alfven velocity and the sound speed in Lagrangian representation.

CHAPTER III. NUMERICAL METHODS

The solutions of the systems (2.48) and (2.50) are studied numerically. Since these systems are of mixed hyperbolic-parabolic type we expect the stability criterion of the difference scheme to depend not only on the sound speed but on other parameters as well. The complexity of the systems makes it doubtful that such a criterion can be found. It is possible, however, to find a stability criterion for the simpler system in which the electron pressure terms are absent. The existence of the energy integral allows us to apply the so called energy method to this case. Having done this we then solve the full system and compare the results with those known to be stable. If the addition of a small amount of electron pressure does not cause serious disturbances in the solution we can assume that stability has not been adversely affected by the addition.

a. The solution without electron pressure

To prove the stability of the finite difference equations used to solve (2.48) with α^- equal zero and α^+ constant we use the energy method. This entails finding a discrete analog of the energy integral (2.54). By determining when the growth of this quantity does not exceed that allowed by the accuracy of the scheme we arrive at a criterion for stability. To obtain this discrete

energy integral it is necessary to make some preliminary changes in the equations of motion so that we are actually solving a slightly altered, though equivalent, system. We first scale the independent variable Y by a factor λ . This allows us to easily change the initial radius of the plasma column. (2.48) is then written in the form

$$\begin{aligned}
 R_{TT} - \frac{1}{\lambda^2} R(B_{3Y})^2 + \frac{1}{2\lambda} (R(B_2^2 + B_3^2)_Y + 2B_2^2 R_Y) + \lambda \alpha^+ (\frac{1}{2} Z_Y + \frac{2Y}{R^3}) &= 0 \\
 (R^2 B_{3Y})_{YT} &= \frac{\lambda}{2} [(R^2)_Y B_3]_T \\
 (3.1) \quad R(RB_2)_{YYT} + \frac{\lambda}{2} \frac{R_T}{R} (R^2)_Y B_2 - \frac{\lambda}{2} (R^2)_{YT} B_2 - \frac{\lambda}{2} (R^2)_Y B_{2T} &= 0 \\
 R_T Z_Y &= \frac{1}{2} (R_Y Z)_T + (R_T Z)_Y .
 \end{aligned}$$

The third equation results from multiplying the second member of (2.48) by R and using some elementary differential identities. As is clear by comparing (3.1) with (2.48), Z equals $1/(YR_Y^3)$. We represent Z by the last equation rather than by its more logical definition because it will be our aim to express the change in the energy integral in terms of the T increment and the explicit appearance of Y makes this impossible. This will be discussed in detail later.

To approximate (3.1) with difference equations the (T,Y) plane is divided into a rectangular lattice with sides parallel to the axes and dividing the T and Y axes into intervals of length k and h respectively. The axes form part of the lattice. To minimize confusion with subscripts

let B represent B_3 and M represent B_2 . Define the difference variables R_j^n , M_j^n and Z_j^n as $R(nk, jh)$, $M(nk, jh)$, and $Z(nk, jh)$ while B_j^n will be $B(nk, (j-1/2)h)$. We will also have to define some difference operators:

$$\begin{aligned}\delta^- A_j^n &= \frac{A_j^n - A_j^{n-1}}{k}, & \delta^+ A_j^n &= \frac{A_j^{n+1} - A_j^n}{k}, & \delta^0 A_j^n &= \frac{A_j^{n+1} - A_j^{n-1}}{2k} \\ \Delta^- A_j^n &= \frac{A_j^n - A_{j-1}^n}{h}, & \Delta^+ A_j^n &= \frac{A_{j+1}^n - A_j^n}{h}, & \Delta^0 A_j^n &= \frac{A_{j+1}^n - A_{j-1}^n}{2h} \\ \delta^2 A_j^n &= \delta^- \delta^+ A_j^n, & \Delta^2 A_j^n &= \Delta^- \Delta^+ A_j^n.\end{aligned}$$

The numerical solution of (3.1) proceeds as follows. From (2.53) and (2.53a) we obtain

$$\begin{aligned}R_j^1 &= R_j^2 = \sqrt{2\lambda_j h}, & M_j^1 &= M_j^2 = m \sqrt{jh}, & Z_j^1 &= Z_j^2 = \frac{1}{jh(\Delta^0 R_j^1)^3} \\ (3.2) \quad B_j^1 &= B_j^2 \quad \text{where} \quad (\Delta^+ B_j^1)^2 - \Delta^+ B_j^{1^2} - \lambda m^2 = 0\end{aligned}$$

$$R_0^n = M_0^n = Z_0^n = 0, \quad B_j^n = g_3(nk), \quad M_j^n = g_2(nk), \Delta^0 R_j^n = 0,$$

for $1 \leq j \leq J$. J is the number of mesh points along the Y axis and is taken to be 400 in our problem. Assume now that we have R , M , B and Z for all j at $n-1$ and n . (3.2) gives this for $n = 2$. R_j^{n+1} is found from the first equation of (3.1) discretized in the form

$$\begin{aligned}(3.3) \quad R_j^{n+1} &= 2R_j^n - R_j^{n-1} + k^2 \left\{ \frac{1}{\lambda^2} R_j^n (\Delta^+ B_j^n)^2 - \frac{1}{2\lambda} R_j^n (\Delta^+ B_j^{n^2} + \Delta^0 M_j^{n^2}) \right. \\ &\quad \left. - \frac{1}{2\lambda} (\Delta^0 R_j^{n^2}) \frac{M_j^{n^2}}{R_j^n} - \lambda \alpha^+ \left[\frac{\Delta^0 Z_j^{n^2}}{4S_j^n} - \frac{2jh}{R_j^n^3} \right] \right\}.\end{aligned}$$

Notice that since R , M and Z behave like the square root of Y better accuracy is obtained by expressing their Y derivatives in the form $\Delta^0 R_j^{n^2} / 2R_j^n$ rather than $\Delta^0 R_j^n$.

Having found R_j^{n+1} we can now find M_j^{n+1} and B_j^{n+1} from

$$(3.4) \quad \begin{aligned} \delta^- \Delta^- (R_j^{n^2} \Delta^+ B_j^n) &= \frac{\lambda}{2} \delta^- [(\Delta^- R_j^{n^2}) B_j^n] \\ R_j^n \delta^0 \Delta^2 (R_j^n M_j^n) &= \frac{\lambda}{2} [M_j^n \Delta^0 \delta^0 R_j^{n^2} + (\delta^0 M_j^n) \Delta^0 R_j^n - \frac{M_j^n}{R_j^n} (\delta^0 R_j^n) (\Delta^0 R_j^{n^2})]. \end{aligned}$$

Both equations in (3.4) represent linear differential equations in the magnetic fields. As difference expressions they become linear inhomogeneous equations for M_j^{n+1} and B_j^{n+1} with the inhomogeneity appearing as linear terms in M_j^n , B_j^n , M_j^{n-1} and B_j^{n-1} . Thus, to an arbitrary solution of the full equation we add that solution of the homogeneous equation which gives the desired value at the boundary.

Since both equations in (3.4) are of second order in Y , before we can apply them we must know the values of the fields at the first two mesh points. The condition of regularity allows us to expand both in power series about the origin. Both equations also admit logarithmic solutions. To find these series expansions let $D = RM$ and write

$$\begin{aligned} R^2(Y, T) &= 2\lambda(A_1(T)Y + A_2(T)Y^2 + A_3(T)Y^3 + A_4(T)Y^4 + \dots) \\ D(Y, T) &= d_1(T)Y + d_2(T)Y^2 + d_3(T)Y^3 + d_4(T)Y^4 + \dots \\ B(Y, T) &= b_0(T) + b_1(T)Y + b_2(T)Y^2 + b_3(T)Y^3 + b_4(T)Y^4 + \dots \\ f(Y, T) &\equiv b_0(0) + b_1(0)Y + b_2(0)Y^2 \dots = f_0 + f_1Y + f_2Y^2 + f_3Y^3 + \dots \end{aligned}$$

The field equations are integrated to give

$$\begin{aligned}
 (R^2 B_Y)_Y - \frac{\lambda}{2} (R^2)_{YB} &= 2\lambda(YF_Y)_Y - \lambda^2 F \\
 (3.5) \quad R D_{YY} - \frac{\lambda}{2} (R^2)_{YD} &= -\sqrt{\frac{\lambda}{2}} m R^2 \\
 (F_Y)^2 + \frac{\lambda}{2} (F^2)_Y - \lambda m^2 &= 0.
 \end{aligned}$$

When the series are substituted into the equations of (3.5) and the coefficients of like powers of Y equated the following set of relations is found.

$$(3.6) \quad \left\{ \begin{aligned} f_1^2 - \lambda f_0 f_1 &= \lambda m^2 \\ 4f_1 f_2 &= 2\lambda f_0 f_2 + \lambda f_1^2 \\ 6f_1 f_3 + 4f_2^2 &= \lambda(3f_0 f_3 + 3f_1 f_2) \\ 8f_1 f_4 + 12f_2 f_3 &= \lambda(f_0 f_4 + 4f_1 f_3 + 2f_2^2) \end{aligned} \right.$$

$$(3.7) \quad \left\{ \begin{aligned} 4d_2 - \lambda d_1 &= -m\lambda\sqrt{2\lambda} \\ 12A_1 d_3 + 4A_2 d_2 - \lambda(A_1 d_2 + 2A_2 d_1) &= -m A_2 \lambda \sqrt{2\lambda} \\ 24A_1 d_4 + 12A_2 d_3 + 4A_3 d_2 - \lambda(A_1 d_3 + 2A_2 d_2 + 3A_3 d_1) \\ &= -m A_3 \lambda \sqrt{2\lambda} \\ 40 A_1 d_5 + 24 A_2 d_4 + 12 A_3 d_3 + 4 A_4 d_2 - \lambda(A_1 d_4 + 2A_2 d_3 \\ &\quad + 3 A_3 d_2 + 4 A_4 d_1) &= -m A_4 \lambda \sqrt{2\lambda} \end{aligned} \right.$$

$$(3.8) \left\{ \begin{array}{l} 2A_1 b_1 - \lambda A_1 b_0 = 2f_1 - f_0 \\ 8A_1 b_2 + 4A_2 b_1 - \lambda(A_1 b_1 + 2A_2 b_0) = 8f_2 - \lambda f_1 \\ 18A_1 b_3 + 12A_2 b_2 + 6A_3 b_1 - \lambda(A_1 b_2 + 2A_2 b_1 + 3A_3 b_0) \\ \quad = 18f_3 - \lambda f_2 \\ 32A_1 b_4 + 24A_2 b_3 + 16A_3 b_2 + 8A_4 b_1 - \lambda(A_1 b_3 + 2A_2 b_2 \\ \quad + 3A_3 b_1 + 4A_4 b_0) = 32f_4 - \lambda f_3 \end{array} \right.$$

(3.7) and (3.8) are used also for the homogeneous difference equations by setting the right hand sides equal to zero. If we knew the values of the A_i 's we could find these coefficients. These are determined from R_j^{n+1} by means of the formula

$$\frac{1}{2\lambda} R_j^{n+1^2} = A_1 jh + A_2 (jh)^2 + A_3 (jh)^3 + A_4 (jh)^4 + O(h^5)$$

in which we let j equal 1, 2, 3 and 4. Thus

$$(3.9) \quad \begin{aligned} A_1 h &= \frac{1}{12\lambda} (24R_1^{n+1^2} - 36R_2^{n+1^2} + 24R_3^{n+1^2} - 6R_4^{n+1^2}) + O(h^5) \\ A_2 h^2 &= \frac{1}{12\lambda} (-26R_1^{n+1^2} + 57R_2^{n+1^2} - 42R_3^{n+1^2} + 11R_4^{n+1^2}) + O(h^5) \\ A_3 h^3 &= -\frac{1}{12\lambda} (3R_1^{n+1^2} - 8R_2^{n+1^2} + 7R_3^{n+1^2} - 2R_4^{n+1^2}) + O(h^5) \\ A_4 h^4 &= \frac{1}{12\lambda} (-R_1^{n+1^2} + 3R_2^{n+1^2} - 3R_3^{n+1^2} + R_4^{n+1^2}) + O(h^5). \end{aligned}$$

There remains now only Z_j^{n+1} to find and this is obtained from

$$(3.10) \quad \delta_{R_j^n}^0 \frac{\Delta^0 Z_j^{n^2}}{Z_j^n} = \frac{1}{2} \delta^- \left(\frac{1}{2} \left(\frac{Z_j^{n+1}}{R_j^{n+1}} + \frac{Z_j^n}{R_j^n} \right) \Delta^0 \frac{R_j^{n+1^2} + R_j^{n^2}}{2} \right) \\ + \Delta^+ (Z_{j-1}^n \delta_{R_j^n}^0 + Z_j^n \delta_{R_{j-1}^n}^0)$$

with Z_1^{n+1} given by $\frac{64 R_1^{n+1^2}}{R_2^{n+1^2}} h^2$ which is a consistent representation for $1/(Y R_Y^3)$ at $Y = h$.

n is replaced by $n-1$ and the process is repeated.

The difference scheme is accurate to second order in h and k .

b. The stability of the system without electron pressure

In finding a stability criterion for the system (3.1) we shall need the definitions

$$(A, B)_\ell = h \sum_{j=\ell}^{J-1} A_j B_j, \quad \|A\|_\ell^2 = (A, A)_\ell$$

and the identities

$$(A, \Delta^- B)_\ell = -(\Delta^+ A, B)_\ell + A_J B_{J-1} - A_{\ell-1} B_{\ell-1}$$

$$(A, \Delta^+ B)_\ell = -(\Delta^- A, B)_\ell + A_{J-1} B_J - A_\ell B_\ell$$

$$(A, \Delta^0 B)_\ell = -(\Delta^0 A, B)_\ell + \frac{1}{2} (A_{J-1} B_J + A_J B_{J-1} - A_\ell B_{\ell-1} - A_{\ell-1} B_\ell).$$

The energy of the system (3.1), E^n , at time n is defined to be

$$\begin{aligned}
E^{n+1} = & \frac{1}{2} \|\delta^- R^n\|_0^2 + \frac{1}{4\lambda} (\|R^n \Delta^+ B^n\|_0^2 + \|R^{n+1} \Delta^+ B^{n+1}\|_0^2) \\
& + \frac{1}{8\lambda} (\|\Delta^+(R^n M^n)\|_0^2 + \|\Delta^+(R^{n+1} M^{n+1})\|_0^2 \\
& + \|\Delta^-(R^n M^n)\|_1^2 + \|\Delta^-(R^{n+1} M^{n+1})\|_1^2) \\
(3.11) \quad & + \frac{1}{8\lambda} (B^n B^{n+1}, \Delta^-(R^{n^2} + R^{n+1^2}))_1 \\
& + \frac{1}{8\lambda} (M^n M^{n+1}, \Delta^0(R^{n^2} + R^{n+1^2}))_1 \\
& + \frac{\lambda}{2} \alpha^+(jh, \frac{1}{R^{n^2}} + \frac{1}{R^{n+1^2}})_0 \\
& + \frac{\lambda \alpha^+}{32} (\frac{Z^n}{R^n} + \frac{Z^{n+1}}{R^{n+1}}, \Delta^0(R^{n^2} + R^{n+1^2}))_0.
\end{aligned}$$

In a manner similar to that used to derive the energy integral (2.54) we find from (3.1) and (3.11)

$$\begin{aligned}
(3.12) \quad & \frac{E^{n+1} - E^n}{K} + \frac{1}{2\lambda} R_{J-1}^n B_J^{n^2} \delta^0 R_{J-1}^n + \frac{1}{4\lambda} (R_J^n M_{J-1}^{n^2} \delta^0 R_J^n + R_{J-1}^n M_J^{n^2} \delta^0 R_{J-1}^n) \\
& - \frac{1}{\lambda^2} B_J^n \delta^0 (R_{J-1}^{n^2} \Delta^- B_{J-1}^n) + \frac{1}{2\lambda^2} (R_J^n M_J^n + R_{J-1}^n M_{J-1}^n) \delta^0 \Delta^-(R_J^n M_J^n) \\
& + \frac{\lambda}{4} \alpha^+(Z_{J-1}^n \delta^0 R_J^n + Z_J^n \delta^0 R_{J-1}^n) = O(k^2).
\end{aligned}$$

A derivation of (3.12) is given in Appendix 2. This equation says that the total change in the energy of the plasma is equal to the energy gained at the boundary plus terms of the order k^2 , which is the accuracy of the difference method. If E^n is a valid energy, that is if it is positive, the difference scheme will be stable.

A sufficient condition that E^n be positive is given by

$$\begin{aligned}
& \sum_{j=1}^{J-1} [(\delta - R_j^n)^2 + \frac{1}{4\lambda} B_j^n B_j^{n-1} \Delta^-(R_j^{n^2} + R_j^{n-1^2}) \\
(3.3) \quad & + (\frac{1}{4\lambda} M_j^n M_j^{n-1} + \frac{\lambda\alpha^+}{16} (\frac{Z_j^n}{R_j^n} + \frac{Z_j^{n-1}}{R_j^{n-1}})) \Delta^0(R_j^{n^2} + R_j^{n-1^2})] \\
& + \frac{\lambda\alpha^+}{16} (\frac{Z_0^n}{R_0^n} + \frac{Z_0^{n-1}}{R_0^{n-1}}) \Delta^0(R_0^{n^2} + R_0^{n-1^2}) > 0.
\end{aligned}$$

The last term in (3.3) is indeterminate as written and is found from the series expansion of R.

Let

$$L_j^n = \frac{1}{4\lambda} M_j^n M_j^{n-1} + \frac{\lambda\alpha^+}{16} (\frac{Z_j^n}{R_j^n} + \frac{Z_j^{n-1}}{R_j^{n-1}}) \text{ and } \Gamma_j^n = \frac{1}{4\lambda} B_j^n B_j^{n-1} + \frac{1}{2} (L_j^n + L_j^{n-1}).$$

Since

$$\begin{aligned}
\sum_{j=1}^{J-1} L_j^n \Delta^0(R_j^{n^2} + R_j^{n-1^2}) &= \sum_{j=1}^{J-1} [\frac{1}{2} (L_j^n + L_{j-1}^n) \Delta^-(R_j^{n^2} + R_j^{n-1^2})] \\
&- \frac{1}{2} L_0^n \Delta^-(R_1^{n^2} + R_1^{n-1^2}) + \frac{1}{2} L_{J-1}^n \Delta^-(R_J^{n^2} - R_J^{n-1^2})
\end{aligned}$$

the condition (3.3) can be written

$$\begin{aligned}
(3.4) \quad & \sum_{j=1}^{J-1} [(\delta - R_j^n)^2 + \Gamma_j^n \Delta^-(R_j^{n^2} + R_j^{n-1^2})] - \frac{1}{2} L_0^n \Delta^-(R_1^{n^2} + R_1^{n-1^2}) \\
& + \frac{1}{2} L_{J-1}^n \Delta^-(R_J^{n^2} + R_J^{n-1^2}) + \frac{\lambda\alpha^+}{16} (\frac{Z_0^n}{R_0^n} + \frac{Z_0^{n-1}}{R_0^{n-1}}) \Delta^0(R_0^{n^2} + R_0^{n-1^2}) > 0.
\end{aligned}$$

The last three terms arise from the presence of the boundaries. Their sum is positive and does not affect the main part of the stability criterion, the summation. We can write this summation in the form

$$(3.15) \sum_{j=1}^{J-1} \left(\frac{1}{2} (\delta^- R_j^n)^2 + \frac{1}{2} (\delta^- R_{j-1}^n)^2 + \Gamma_j^n \Delta^- (R_j^{n^2} + R_j^{n-1^2}) \right) \\ - \frac{1}{2} (\delta^- R_0^n)^2 + \frac{1}{2} (\delta^- R_{J-1}^n)^2 > 0 .$$

The last term is positive and $\delta^- R_0^n$ is zero. Writing $R_j^{n^2}$ as $R_j^{n-1^2} + (R_j^n + R_j^{n-1}) \delta^- R_j^n$ we see that each term in the series in (3.15) will be positive if

$$(3.16) \frac{1}{2} (\delta^- R_j^n)^2 + \frac{1}{2} (\delta^- R_{j-1}^n)^2 + \Gamma_j^n \Delta^- (2R_j^{n-1^2} + (R_j^n + R_j^{n-1}) \delta^- R_j^n) > 0$$

for $0 < j < J$. A sufficient condition that (3.16) holds is

$$(3.17) \frac{1}{2} (\delta^- R_j^n)^2 + \frac{1}{2} (\delta^- R_{j-1}^n)^2 + 2\Gamma_j^n \Delta^- R_j^{n-1^2} > | \Gamma_j^n \Delta^- ((R_j^n + R_j^{n-1}) \delta^- R_j^n) |_k .$$

However, the right hand side of this satisfies

$$k | \Gamma_j^n \Delta^- ((R_j^n + R_j^{n-1}) \delta^- R_j^n) | \leq \frac{k}{h} | \Gamma_j^n | (| (R_j^n + R_j^{n-1}) \delta^- R_j^n | \\ + | (R_{j-1}^n + R_{j-1}^{n-1}) \delta^- R_{j-1}^n |) \leq \frac{1}{2} \frac{k^2}{h^2} \Gamma_j^{n^2} ((R_j^n + R_j^{n-1})^2 \\ + (R_{j-1}^n + R_{j-1}^{n-1})^2) + \frac{1}{2} (\delta^- R_j^n)^2 + \frac{1}{2} (\delta^- R_{j-1}^n)^2 .$$

Upon substituting this into (3.17) we arrive, after cancellation, at the stability criterion

$$(3.18) \left(\frac{h}{k} \right)^2 > \frac{\Gamma_j^n [(R_{j-1}^n + R_{j-1}^{n-1})^2 + (R_j^n + R_j^{n-1})^2]}{4 \Delta^- R_j^{n-1^2}} .$$

Since the sound and Alfven speeds are given by

$$C_S^2 = \frac{3\lambda\alpha^+}{2YR_Y^4} \quad \text{and} \quad C_A^2 = \frac{R(B^2 + M^2)}{\lambda R_Y}$$

the stability criterion (3.18) specifies, discounting the centering of the quantities in the mesh, that

$$\left(\frac{h}{k}\right)^2 > \frac{1}{4} C_A^2 + \frac{1}{12} C_S^2.$$

A stability check is performed at each step in the numerical calculations using (3.18). Also, the energy equation (2.54) is integrated numerically as a check on the accuracy of the calculations. The two parts of the integral still agree to better than five parts in a thousand at time $T = 20$.

We are now in a position to justify the use of the system (3.1) in solving this problem rather than using (2.48) as it stands. Whereas the equation for B lends itself readily to incorporation into the discrete energy equation, it was found that the equation for M did not. The equation for M that appears in (3.1) was rigged in such a manner as to make all troublesome terms cancel in the final addition. For this convenience we pay a small price in the complexity of the equation. The trouble with leaving Z as $1/YR_Y^3$ appears when the energy equation is derived in discrete form to prove stability. The object is to make the change in energy in one time step proportional to k^3 . However, we find that in the derivation we use differential identities in which Y appears. Such terms can be correct only to $O(h^2)$. Instead of having the change in energy $O(k^3)$ we could obtain it only to $O(h^2 + k^3)$.

and thus not sufficient for proving stability. By using the differential equation to define $1/YR_Y^3$ we have eliminated this explicit dependence on Y and have forced the energy equation into the desired form. Of course, we have had to introduce another variable and another equation.

c. Numerical simulation of the pinch with pressure

The inclusion of the electron pressure causes considerable complication of the numerical method used to solve the problem. This arises primarily from the time derivatives which appear on the right hand sides of the field equations. While it seems possible to solve the new system numerically we have not been able to devise a stable method. However, by considering terms of $O(\alpha^{-2})$ negligible a stable and greatly simplified method can be used.

Again, scaling the Y variable by the parameter λ , (2.48) becomes

$$\begin{aligned}
 R_{TT} - \frac{1}{\lambda^2} R(B_{3Y})^2 + \frac{1}{\lambda} B_2(RB_2)_Y + \frac{1}{\lambda} RB_3B_{3Y} \\
 + \left(\frac{\alpha^+\lambda}{2YR_Y^3}\right)_Y - \frac{2\lambda\alpha^+Y}{R^3} + \left(\frac{\alpha^-P^2}{R_Y}\right)_Y - \frac{\alpha^-\lambda^2}{R^3R_Y^2P^2S^2} = 0 \\
 (3.19) \quad H_{2T} = \frac{5\lambda^2}{12} \left(\frac{\alpha^-S^2T}{H_2}\right)_{YY} ; \quad H_{3T} = -\frac{64\lambda^2}{27\pi} \left(\frac{\alpha^-H_3S^2T}{H_2^2}\right)_{YY} ; \\
 H_2 + \frac{1}{\lambda} R_YB_2 - \frac{1}{\lambda^2} (RB_2)_{YY} ; \quad H_3 = \frac{1}{\lambda} RR_YB_3 - \frac{1}{\lambda^2} (R^2B_{3Y})_Y ;
 \end{aligned}$$

$$P^6 = \frac{\lambda^3}{R} - \frac{\lambda B_3 - 2B_{3Y}}{H_3} \frac{B_2}{H_2} ; \quad S^6 = \frac{1}{RR_Y} - \frac{\lambda B_3 - 2B_{3Y}}{H_3} \left(\frac{H_2}{B_2} \right)^2 .$$

The equations for H_2 and H_3 in (3.19) can be written in the form

$$(3.20) \quad \begin{aligned} H_{2T} &= \frac{5\lambda^2}{12} \left[\alpha^- \left(\frac{S^2}{H_2} \right)_T + \alpha^- \frac{S^2}{H_2^2} H_{2T} \right]_{YY} \\ H_{3T} &= - \frac{64\lambda^2}{27\pi} \left[\alpha^- \left(\frac{H_3 S^2}{H_2^2} \right)_T - \alpha^- \frac{S^2}{H_2^2} H_{3T} + \alpha^- \frac{2H_3 S^2}{H_2^2} H_{2T} \right]_{YY} . \end{aligned}$$

However, since H_{2T} and H_{3T} are $O(\alpha^-)$ these equations give

$$(3.21) \quad \begin{aligned} H_{2T} &= \frac{5\lambda^2}{12} \left(\alpha^- \frac{S^2}{H_2} \right)_{TYY} + O(\alpha^{-2}) \\ H_{3T} &= - \frac{64\lambda^2}{27\pi} \left(\alpha^- \frac{H_3 S^2}{H_2^2} \right)_{TYY} + O(\alpha^{-2}) . \end{aligned}$$

Neglecting the higher order terms these equations can be integrated with respect to T using the initial conditions given in (2.53). Further simplification can be achieved if we notice that H_2 and H_3 , as well as all Y derivatives, remain within $O(\alpha^-)$ of their initial values, denoted by H_{20} and H_{30} . Since H_2 , H_3 and S are initially constant (3.21) when integrated yields

$$(3.22) \quad \begin{aligned} \tilde{H}_2 &= H_{20} + \frac{5\lambda^2}{12} \frac{1}{H_{20}} (\alpha^- S^2)_{YY} = H_2 + O(\alpha^{-2}) \\ \tilde{H}_3 &= H_{30} - \frac{64\lambda^2}{27\pi} \frac{H_{30}}{H_{20}^2} (\alpha^- S^2)_{YY} = H_3 + O(\alpha^{-2}) . \end{aligned}$$

Since we are solving this problem to first order in α^- we see that it is only necessary to find P and S to

this order since whenever they appear in (3.19) they are multiplied by a factor α^- . We therefore introduce an error of $O(\alpha^{-2})$ by using

$$(3.23) \quad \tilde{P}^6 = \frac{1}{R} \frac{B_3 - \frac{2}{\lambda} B_{3Y}}{H_{30}} \frac{B_2}{H_{20}} \quad ; \quad \tilde{S}^6 = \frac{1}{RR_Y} \frac{B_3 - \frac{2}{\lambda} B_{3Y}}{H_{30}} \left(\frac{H_{20}}{B_2} \right)^2$$

in place of P and S in (3.19) and (3.22). We thus obtain from (3.19) the following system which differs from it only by terms of $O(\alpha^{-2})$.

$$(3.24) \quad \begin{aligned} R_{TT} - \frac{1}{\lambda^2} R(B_{3Y})^2 + \frac{1}{\lambda} B_2 (RB_2)_Y + \frac{1}{\lambda} RB_3 B_{3Y} \\ + \lambda \left(\frac{\alpha^+}{2Y R_Y} \right)_Y - \frac{2\lambda \alpha^+ Y}{R^3} + \left(\frac{\alpha^- \tilde{P}^2}{R_Y} \right)_Y - \frac{\alpha^- \lambda^2}{R(RR_Y \tilde{P} S)^2} = 0 \\ \tilde{H}_2 = H_{20} + \frac{5\lambda^2}{12H_{20}} (\alpha^- S)_{YY} \\ \tilde{H}_3 = H_{30} - \frac{64\lambda^2}{27\pi} \frac{H_{30}}{H_{20}^2} (\alpha^- S)_{YY} \\ (RB_2)_{YY} - \lambda R_Y B_2 + \lambda^2 \tilde{H}_2 = 0 \\ (R^2 B_{3Y}) - \lambda RR_Y B_3 + \lambda^2 \tilde{H}_3 = 0 . \end{aligned}$$

By setting B_3 and $\tilde{H}_3$ equal to zero in (3.24) we obtain the equations used in the numerical simulation of the z pinch. For the theta pinch the system must be modified to correspond to (2.50). When this is done we obtain a new equation for $\tilde{H}_3$ given by

$$\tilde{H}_3 = H_{30} - \frac{25}{48\pi} \frac{1}{H_{30}} \left(\frac{\alpha^-}{P^2 R_Y^2} \right)_{YY} .$$

B_2 and $\tilde{H}_2$ are set equal to zero in the remaining equations.

The numerical solution of this system is now not unlike that of the one in which the electron pressure is absent. Assuming we know all the variables at two time steps we find R at the next time level from the first equation in (3.24). We then find B_2 and B_3 using the initial values of H_2 and H_3 . This will, of course, be correct only to $O(\alpha^-)T$, but that is sufficient to find the new values of $\tilde{P}$ and $\tilde{S}$ since they are desired only to this accuracy. With the new $\tilde{P}$ and $\tilde{S}$ we find $\tilde{H}_2$ and $\tilde{H}_3$ and from these we obtain corrected values of B_2 and B_3 . The procedure is now repeated to advance in time.

Although we have no proof of the stability of this method we may infer from the well behaved results and the results of similar problems for which stability has been proven that the method is stable.

CHAPTER IV. NUMERICAL SIMULATION OF THE PINCH

The systems (2.48) and (2.50) with the initial and boundary conditions given in (2.53) and (2.53a) are solved numerically by the methods described in Chapter III. The results of those computations are presented in this chapter.

a. The pure theta pinch

System (2.50) representing a theta pinch is solved under the following conditions. To a column of plasma of radius ten satisfying the equilibrium conditions given in (2.53) we raise B_z linearly at the boundary to double its initial value in a time T_f . Thereafter B_z is held at two. The parameter T_f regulates the strength of the compression by specifying the length of time the plasma feels the full strength of the field. Figures 1, 2 and 3 show the density profile of the plasma column for pressure-free cases in which T_f equals 10, 7 and 2 respectively. In all three we see a train of waves being generated by the boundary and travelling inward ahead of the incoming plasma. Upon reaching the origin the density rises to several times its initial value. The waves are then reflected outward where they interact with the remaining tail of the wave train. In Figure 1, showing the mildest compression, the first wave reaches the origin at approximately $T = 10.1$ and the density rises to a value of about 5.6. Since our scaling velocity

is the Alfven velocity and our initial radius is ten we see that the wave travels at approximately the Alfven speed, as a linearized theory would predict. In Figure 2, for which $T_f = 7$, the first wave reaches the origin at $T = 9.8$ and the density rises there to 9.4. In the much stronger compression produced by $T_f = 2$ and shown in Figure 3 the disturbance arrives at the center at $T = 7.2$ causing a density rise in excess of 30. Notice that in Figure 3 when the first wave is reflected from the origin it draws with it most of the gas so the density falls rather sharply at the boundary. When the second wave reaches the center at $T = 11$ the density is seen to again rise quite strongly. Finally, we see the first wave has traveled back through the plasma and has been reflected at the outer boundary. Since the plasma as a whole is now moving outward the reflected wave gains in strength. At $T = 14$ a good part of the plasma is concentrated in this wave.

Figure 4 shows the effect of pressure on the motion. Again taking $T_f = 7$ but now setting $\alpha^+ = \alpha^- = .04$ we see by comparing Figure 4 with its pressure-free counterpart Figure 2 that the pressure causes two obvious changes in the motion. In addition to the initial Alfven speed of unity we now have an initial sound speed which equals $\sqrt{3\alpha^+ + 3\alpha^-}$, or $\sqrt{.24}$. Thus instead of the disturbance taking nearly 10 time units to reach the center it now takes about $R_M / \sqrt{C_A^2 + C_S^2} = 10 / \sqrt{1 + .24} \approx 8.96$. In the actual numerical simulation the first wave

reaches the origin at about 8.82. In addition to the increase in the speed of propagation we observe another effect, the tendency of the waves to be more sharply peaked. The pulses are no longer rounded as they were in Figure 2 but have a steeper front and sharper peak. When the pulse reaches the origin the density rises above the value 17, nearly twice that of the pressure-free case with its broader, more rounded wave. The effect of increasing α^+ and α^- is to cause the wave fronts to steepen until the density becomes discontinuous.

The axial magnetic field is approximately equal to the density, differing from it only in that the pulses tend to be weaker and lag slightly behind those of the density. These deviations from the one-fluid fields are produced by the terms in H_3 involving derivatives of B_3 .

Kilb (12) has obtained similar results in theta pinch calculations.

b. The pure z pinch

Corresponding to equations (2.51) and (2.52) is a compression in which the only field present is B_2 . Unless we take a non-constant initial pressure the only equilibrium is one in which B_2 is identically zero. If we start the problem with no magnetic field we have to define a new scaling parameter in (2.4). We take for this the final value of the magnetic field at the boundary. Thus, the

boundary conditions on B_2 for this problem are

$$(4.1) \quad B_2(T, Y_M) = \begin{cases} \frac{T}{T_f} & , \quad 0 \leq T \leq T_f \\ 1 & , \quad T_f \leq T . \end{cases}$$

We find that with this pinch it is impossible, even with very mild compressions, to obtain solutions which do not break almost immediately. The reason for this is evident. Without a magnetic field in the plasma the only mode of propagation initially present is the pressure. Thus, until the field penetrates into the plasma, the behavior is similar to an ordinary fluid, that is, discontinuities tend to develop upon compression. Since we are concerned in this paper with well behaved flows we shall not pursue the investigation of the pure z pinch but rather examine a hybrid pinch in which a magnetic field is initially present.

c. A hybrid z -theta pinch

In this pinch we start with the same equilibrium as in the theta pinch, both N and B_3 equal to one. In this case, however, the motion is produced by raising B_2 at the boundary while imposing the condition that B_{3Y} vanishes there. With this condition on B_{3Y} no axial component of the field enters or leaves the plasma at the boundary and we shall show that the total flux of B_3 in the gas is constant to $O(\alpha^-)$. The time derivative of this flux is

$$\begin{aligned}
R_M(T) \int_0^{R_M(T)} r B_3(t, r) dr &= \frac{d}{dT} \int_0^{Y_M} R R_Y B_3 dY = \frac{d}{dT} \int_0^{Y_M} H_3 dY + (R^2 B_{3Y})_Y dY \\
&= \frac{d}{dT} \left\{ \int_0^{Y_M} H_3 dY + R^2 B_{3Y} \Big|_0^{Y_M} \right\} .
\end{aligned}$$

We have used the definition of H_3 given in (2.47). By virtue of the boundary conditions the second term vanishes while the integral can be evaluated using (2.48) to yield

$$\begin{aligned}
\int_0^{Y_M} H_{3T} dY &= - \frac{64\alpha^-}{27\pi} \int_0^{Y_M} \left(\frac{H_3 S^2 T}{H_2^2} \right)_{YY} dY \\
&= - \frac{64\alpha^-}{27\pi} \left(\frac{H_3 S^2 T}{H_2^2} \right)_Y \Big|_{Y=0}^{Y=Y_M} .
\end{aligned}$$

Thus,

$$(4.2) \quad R_M(T) \int_0^{R_M(T)} r B_3(t, r) dr = - \frac{64\alpha^-}{27\pi} \left(\frac{H_3 S^2 T}{H_2^2} \right)_Y \Big|_{Y=0}^{Y=Y_M}$$

giving us the desired result.

B_2 is raised linearly at the boundary from its initial value of 0 to a value of .6 in a time $T_f = 10$. Thereafter it is held fixed at .6. To allow the computations to proceed without the solution breaking we take the initial pressure to be quite small: $\alpha^+ = \alpha^- = .001$. Because of the high densities encountered in this case the pressure, even with this small initial value, becomes significant.

In Figure 5a we observe that this pinch takes longer to develop than a theta pinch. In the latter the disturbance has reached the origin at $T = 10$ while in this pinch it has propagated only halfway to the origin. This delay occurs because the magnetic field enters the momentum equation quadratically and whereas the compressing field was initially one in the theta pinch it is initially zero in this pinch. Thus, it takes a while before sufficient B_2 is present. Once the motion does begin, however, it is much stronger than anything observed in the theta pinch. One reason for the rapid compression is that as the density rises near the boundary so does B_3 and this aids in the initial compression stages. A more fundamental difference is that in the z pinch there is no equilibrium state nearby about which the plasma column can oscillate. As we saw in the theta pinch, the radius does not continue to diminish but rather oscillates about a near-equilibrium state. Indeed, with any dissipation present the plasma would eventually approach an equilibrium of constant density and constant magnetic field. In the z pinch, though, the plasma is not close to an equilibrium state and so the radius continues to fall. To stop the collapse it would be necessary to take into account the jump conditions at the plasma-vacuum boundary where the pressure will eventually become strong enough to balance the field. Since we have ignored this effect our solution would soon become singular. These considerations, however, do not affect

the validity of the problem in the range illustrated in Figure 5.

Referring now to Figure 5b, we see that by $T = 12$ the radius of the column has shrunk to 4.9. At $T = 12.8$ the first pulse reaches the origin where it peaks to a value greater than 35. At $T = 13$ this pulse has been reflected outward toward the boundary which is now at a distance of only 3.5 away from the origin and still moving inward. Finally, by $T = 14$ the radius has fallen below 2 meaning that the average density is greater than 30. During the final stages of the collapse, local density values of near 100 are observed.

The noteworthy characteristic of this flow is that very strong compressions and large density rises are obtained without initiating the formation of discontinuities even though the value of B_z which must be imposed on the boundary to satisfy the condition $B_{zy} = 0$ reaches above 20. In a theta pinch it would be impossible to increase B_z to this value so quickly without the solution breaking down. Part of the reason for the well behaved nature of the solution is that the value of B_z at the boundary is adjusted in such a way that there is no force produced by this component at the boundary.

If the initial pressure is taken to be larger, discontinuities in the density develop. Even with fairly substantial initial pressure, though, the solution remains well behaved until large densities are reached. For instance,

with an initial pressure of $\alpha^+ = \alpha^- = .05$ the radius still falls below 4.5 without the solution breaking. If an artificial viscosity is added it appears that the smoothness of the solution can be maintained and the shock resolved. It should be mentioned that the ratio $\beta = (P_{11}^+ + P_{11}^-) / \frac{1}{2} B^2$ remains small even though the pressure increases to many times its initial value because B_z^2 , like the pressure, is approximately proportional to the density squared.

d. Other flows

When two components of the magnetic field are present in the plasma we observe, by special choice of the boundary conditions, unusual phenomena which arise from the distortion of one component by the compression initiated by the other component. These effects depend strongly on the initial geometry of the fields and can become so strong as to prevent the numerical computations from proceeding.

We take the pressure to be zero in all these cases. The equilibrium is determined by the condition that forces produced by the fields balance. Thus, the appropriate initial conditions are given in (2.53a). From this equilibrium one of the magnetic fields is raised at the boundary while the value of the other field at the boundary is held fixed. It is this condition which produces the unusual flows.

In Figure 6 we show the density profile which results

from raising B_2 while keeping B_3 equal to zero on the boundary. The initial fields are shown in the upper left hand corner of the graph. Figure 6 should be compared with Figure 1 in which B_3 was raised at the same rate. Two differences are seen immediately. First, the waves are weaker and broader. The most striking difference, though, is the sharp rise in density at the boundary which occurs in Figure 6 but is absent in Figure 1. This effect is more pronounced in Figure 7 in which more B_3 and less B_2 is present initially. Again, the total field strength at the boundary starts at one and is raised linearly to two and then held fixed. If we start with the equilibrium in which no B_2 is present, that is as we did in the pure theta pinch, but this time raise B_2 we find that for the same strength compression the density at the boundary becomes so large so fast that the numerical computations fail.

To understand this phenomenon we must look at the geometry of the fields and the imposed boundary conditions. In Figure 11 we have shown the density and two magnetic fields for the problem considered in Figure 7, that is, starting from the equilibrium in which B_2 and B_3 are equal on the boundary we raise B_2 . As B_2 is increased at the edge the plasma moves inward and the density near the boundary rises. B_3 , behaving like the density also rises near the boundary. The value of B_3 at the boundary is, however, held at its initial value. This creates an increased B_3 inside

the plasma which tends to inhibit the inward motion of the gas at the boundary. Thus the gas near the boundary is caught between the rising B_2 field at the boundary and the rising B_3 field inside. This gives rise to the strong increase in density at the boundary. These events can be seen in Figure 11. If the amount of B_3 present is not too large the compression wave can overcome the barrier and propagate into the plasma. When this happens the B_3 peak also propagates inward, temporarily relieving the pressure. This can be seen in Figure 6 at times 5 and 9 and in Figure 7 at times 6 and 9. In the case where there is no B_2 present initially, though, the density has already risen past computable values before the B_3 barrier can propagate inward and relieve the pressure. Because any compression waves that do propagate inward must first overcome this resistance they are weaker than their counterparts in the pure theta pinch.

There are other factors which contribute to the rapid rise in density produced when B_2 is used to compress the plasma. These are related to the means of propagation which are available to a disturbance. Since there is no pressure in this problem the only means available is through magnetic compression waves. These are produced by the magnetic fields acting as pressures in the momentum equation

$$(4.3) \quad R_{TT} + \frac{1}{2} R(B_2^2 + B_3^2)_Y = R(B_{3Y})^2 - R_Y B_2^2 .$$

The field terms on the left are the magnetic pressure terms while those on the right are related to the cylindrical geometry and are in general much smaller away from the origin. If we attempt to compress the plasma using B_2 and start from an equilibrium in which no B_2 is present we see that since B_2 appears in (4.3) only quadratically it is very inefficient as a means of wave propagation. Since B_3 is held fixed at the boundary it cannot help much in propagating waves inward from the edge. Once inside the plasma, though, B_3 becomes an effective means of wave propagation since it is no longer restricted. In fact, near the origin it becomes the only means of wave propagation since B_2 must vanish there. If we provide another means of propagation by adding ion pressure to the problem the peaking is reduced.

There is another boundary phenomenon which occurs when both fields are present. If we again start from that equilibrium in which B_2 is one at the boundary while B_3 vanishes there and raise B_3 instead of B_2 we obtain a very different result. Whereas when B_2 was raised, as illustrated in Figure 5, the density rose at the boundary, when B_3 is raised it falls! This is shown in Figure 8. Notice that this effect is quite strong with the density at the boundary falling to about one-tenth of its initial value.

The explanation for this behavior again lies in the very special geometry of the initial fields. Initially,

to balance the force produced by the positive constant slope of B_2 , B_3 has a negative slope. B_3 , however, contributes two terms to the equilibrium, the magnetic pressure $B_3 B_{3Y}$ and the centrifugal term $(B_{3Y})^2$. In general, the pressure term is much larger but in this case we have made B_3 vanish at the boundary and so the centrifugal term alone must provide the force needed to maintain equilibrium. This gives rise to a large negative slope for B_3 at the boundary. When B_3 is raised at the edge two things happen. Where before there was no B_3 , and thus no magnetic pressure, there now is magnetic pressure and since the slope of B_3 is large and negative the effect of the pressure is to drive the plasma outward. Thus, even though we are increasing B_3 , the effect at the boundary is to push the gas away from the center. There is another effect caused by the increasing B_3 which also contributes to the density drop. As the column as a whole contracts under the increasing B_3 field the density inside rises, and so, therefore, does B_2 . Since B_2 is fixed at one at the boundary the rise of B_2 inside the plasma has the effect of diminishing the slope of B_2 at the boundary. Thus, the ability of B_2 to contain the plasma is diminished at the edge as the B_3 field grows. The density at the boundary thus falls. These developments are illustrated in Figure 10. Notice in particular how the slope of B_2 is diminished from its positive initial value and how, by $T = 10$, it has become negative, thus

contributing to the outward acceleration of the plasma at the boundary.

This density drop is a direct consequence of the fact that B_z vanishes initially at the boundary. For when B_z is initially present in any non-trivial amount this drop does not occur. Figure 9 shows the density profile produced by raising B_z from the initial state in which both fields are equal at the boundary. The drop in density has disappeared and we now even see a slight rise in the density at the boundary caused by the same mechanism described earlier with the role of the fields reversed. By increasing the initial value of B_z still further we approach the pure theta pinch of Figure 1.

Since the boundary effects discussed in this section are primarily local phenomena they depend on the initial radius of the column only insofar as it affects the local field geometry. Thus, these phenomena are not associated with cylindrical compression but arise whenever one attempts to compress a plasma which contains a magnetic field which is held fixed at the boundary.

APPENDIX 1. Characteristics

In this section we derive the characteristic equations given in (2.54). We examine first the simple case in which there is no pressure. (2.48) then becomes

$$R_{TT} - R(B_3 Y)^2 + B_2(RB_2)_Y + RB_3 B_3 Y = 0$$

$$(A1.1) \quad [(RB_2)_{YY} - R_Y B_2]_T = 0$$

$$[(R^2 B_3)_Y - RR_Y B_3]_T = 0 .$$

From the second two equations we immediately obtain the characteristics $(dY)^2 = 0$. After integrating we can write the equations as

$$R_{TT} + \dots = 0$$

$$(A1.2) \quad RB_{2YY} + B_2 R_{YY} + \dots = 0$$

$$R^2 B_{3YY} + \dots = 0$$

where the dots refer to terms with lower order derivatives.

The last equation gives the characteristics $(dT)^2 = 0$.

We are thus left with

$$R_{TT} + \dots = 0$$

(A1.3)

$$RB_{2YY} + B_2 R_{YY} + \dots = 0$$

which can easily be shown to contribute the characteristics

$(dT \, dY)^2 = 0$. Thus the full characteristic equation is

$dT^4 \, dY^4 = 0$. For the pure z pinch this becomes $dT^2 \, dY^3 = 0$

and for the pure theta pinch we obtain $dT^2 dY^3 = 0$.

The inclusion of ion pressure changes the first equation to

$$R_{TT} - R(B_{ZY})^2 + B_2(RB_2)_Y + RB_3B_{ZY} + \left(\frac{\alpha^+}{2YR_Y}\right)_Y - \frac{2\alpha^+_Y}{R^3} = 0$$

while leaving the others unchanged. After integrating and obtaining the characteristics $(dY)^2 = 0$ the system becomes

$$\begin{aligned} R_{TT} - c_L^+ R_{YY} + \dots &= 0 \\ (A1.4) \quad RB_{2YY} + B_2 R_{YY} + \dots &= 0 \\ R^2 B_{3YY} + \dots &= 0 . \end{aligned}$$

From the last equation we again obtain $(dT)^2 = 0$ while the first two yield $dT^2(dY^2 - c_L^2 dT^2) = 0$. Thus the characteristic equation for the system is

$$dT^6 dY^2 \left[\left(\frac{dY}{dT} \right)^2 - c_L^{+2} \right] = 0 .$$

The characteristic equation for both the z and theta pinches in this case is given by

$$dT^4 dY \left[\left(\frac{dY}{dT} \right)^2 - c_L^{+2} \right] = 0 .$$

We consider now the problem in which both the electron and ion pressures are included, examining first the pure theta pinch. The system (2.50) can be written in the abbreviated form

$$\begin{aligned}
R_{TT} - c_L^{+2} R_{YY} - \frac{\alpha^- P^2}{R_Y^2} R_{YY} + \frac{\alpha^-}{R_Y} P_{YY}^2 + \dots = 0 \\
(A1.5) \quad R \left[\frac{\alpha^- P^2}{R_Y} T_{YY} - \frac{2\alpha^- P^2}{R_Y^2} R_{TTY} \right] + \dots = 0, \text{ where} \\
P_Y^4 = \frac{B_Z - 2B_{ZY}}{RR_Y H_Z^2} R^2 B_{ZYY} + \dots
\end{aligned}$$

From the first two equations we obtain the characteristic equation

$$dT^2 dY [dY^2 - (c_L^{+2} + c_L^{-2}) dT^2] .$$

However, we wish to solve for B_Z and R , not P and R so the characteristic condition for the pure theta pinch becomes

$$dT^4 dY [dY^2 - (c_L^{+2} + c_L^{-2}) dT^2] = 0.$$

The system (2.52) representing the pure z pinch can be written in the form

$$\begin{aligned}
R_{TT} - c_L^{+2} R_{YY} - \frac{\alpha^- P^2}{R_Y^2} R_{YY} + \frac{\alpha^-}{R_Y} P_{YY}^2 + \dots = 0 \\
(A1.6) \quad S_{TTY}^2 + \dots = 0 \quad \text{but now } P_Y \text{ is given by} \\
P_Y^4 = - \frac{R_Y B_Z}{(RR_Y)^2} \frac{1}{H_Z^2} (RB_{ZYY} + B_Z R_{YY}) .
\end{aligned}$$

With this definition of P_Y we see that the third derivatives of R now appear in the first equation, making the P_Y^2 term dominate. However, we see that the dominating parts of the first two equations are equal. We must therefore eliminate P in the first equation using the second to obtain an equation

with independent dominating terms. This is done in the following manner. Using $PS = 1/(RR_Y)$ the system becomes

$$R_{TT} - c_L^{+2} R_{YY} - \frac{\alpha^- P^2}{R_Y^2} R_{YY} - \frac{\alpha^-}{R_Y} \left(\frac{1}{RR_Y S} \right)^2 \left(\frac{2R_{YY}}{R_Y} + \frac{S^2}{S^2} \right) + \dots = 0$$

$$(A1.7) \quad S_{TYY}^2 - 2 \frac{H_{2Y}}{H_2} S_{TY}^2 - \frac{H_{2YY}}{H_2} S_T^2 + \dots = 0 \quad \text{where}$$

$$S_Y = \frac{S}{P} P_Y + \dots \quad \text{and } P_Y \text{ is given above.}$$

We now differentiate the first equation, remembering that this adds characteristics which must later be eliminated.

This gives

$$\begin{aligned} R_{TTY} - c_L^{+2} R_{TTY} - \frac{\alpha^- P^2}{R_Y^2} R_{TTY} \\ - \frac{\alpha^-}{R_Y} \left(\frac{1}{RR_Y S} \right)^2 \left(-\frac{2R_{TTY}}{R_Y} + \frac{S^2}{S^2} \right) + \dots = 0. \end{aligned}$$

When we use the second equation to eliminate S_{TYY} we are left with the system

$$\begin{aligned} R_{TTY} - (c_L^{+2} + c_L^{-2}) R_{TTY} - \frac{2\alpha^- P^2}{R_Y} \frac{H_{2Y}}{H_2} S_{TY}^2 - \frac{\alpha^- P^2}{R_Y} \frac{S_T^2}{H_2} H_{2YY} + \dots = 0 \\ S_{TYY}^2 + \dots = 0. \end{aligned} \quad (A1.9)$$

These two equations are independent and to find the characteristics we use the definitions of H_2 and S . Thus

$$S_Y^4 = -\frac{1}{(RR_Y)^2} \frac{1}{R_Y M} (RB_{2YYY} + B_{2YYY}) + \dots \quad \text{and}$$

$$H_{2Y} = -RB_{2YYY} - B_{2YYY} + \dots$$

We can see by observation that since the principle parts of H_2 and S are proportional the last two terms in the first equation of (A1.9) will contribute nothing to the characteristics by virtue of the second equation. The characteristics for the system are thus found to be

$$dT^4 dY [dY^2 - (c_L^{+2} + c_L^{-2}) dT^2] = 0 .$$

Determining the characteristics of system (2.48), which contains both magnetic fields, requires the same type of elimination procedure used for system (2.52) only now we have three equations with equal highest order terms. Omitting the details we simply state the result. The characteristic equation for (2.48) is

$$dT^6 dY^2 [dY^2 - (c_L^{+2} + c_L^{-2}) dT^2] = 0 .$$

APPENDIX 2. The Discrete Energy Equation

We now derive the energy equation (3.11) associated with the difference equations used to solve (3.1).

The system (3.1) written as the difference equations (3.3), (3.4), and (3.10) is

$$\begin{aligned}
 & \delta^2 R_j^n - \frac{1}{\lambda^2} R_j^n (\Delta^+ B_j^n)^2 + \frac{1}{2\lambda} R_j^n (\Delta^+ B_j^{n^2} + \Delta^0 M_j^{n^2}) \\
 & + \frac{1}{2\lambda} \frac{M_j^{n^2}}{R_j^n} \Delta^0 R_j^{n^2} + \alpha^+ \left[\frac{\Delta^0 Z_j^{n^2}}{4Z_j^n} - \frac{2jh}{R_j^n} \right] = 0 \\
 & R_j^n \delta^0 \Delta^2 (R_j^n M_j^n) - \frac{\lambda}{2} (M_j^n \Delta^0 \delta^0 R_j^{n^2} + \delta^0 M_j^n \Delta^0 R_j^{n^2} \\
 (A2.1) \quad & - \frac{M_j^n}{R_j^n} (\delta^0 R_j^n) (\Delta^0 R_j^{n^2})) = 0 \\
 & \delta^- \Delta^- (R_j^{n^2} \Delta^+ B_j^n) - \frac{\lambda}{2} \delta^- (B_j^n \Delta^- R_j^{n^2}) = 0 \\
 & \delta^0 R_j^n \frac{\Delta^0 Z_j^{n^2}}{Z_j^n} - \frac{1}{8} \delta^- \left[\left(\frac{Z_j^{n+1}}{R_j^{n+1}} + \frac{Z_j^n}{R_j^n} \right) \Delta^0 (R_j^{n+1^2} + R_j^{n^2}) \right] \\
 & - \Delta^+ (Z_{j-1}^n \delta^0 R_j^n + Z_j^n \delta^0 R_{j-1}^n) = 0 .
 \end{aligned}$$

In deriving the energy equation we follow the procedure used to obtain (2.54). Thus, the first equation in (A2.1) is multiplied by $\delta^0 R_j^n$, the second equation is multiplied by $-\frac{1}{\lambda^2} M_j^n$, and the third multiplied by $-\frac{1}{\lambda^2} B_j^n$. The term by term results of these multiplications are

$$\delta^0_{R_j} \delta^2_{R_j} = \frac{1}{k} \left[\frac{1}{2} (\delta^{-R_j^{n+1}})^2 - \frac{1}{2} (\delta^{-R_j^n})^2 \right]$$

$$\frac{-1}{\lambda^2} R_j^n \delta^0_{R_j} (\Delta^+ B_j^n)^2 = - \frac{1}{4\lambda^2} \delta^- (R_j^{n+1} \Delta^+ B_j^{n+1} - R_j^n \Delta^+ B_j^n)^2$$

$$- \frac{1}{2\lambda^2} \Delta^- B_j^n \delta^- [(\Delta^+ B_j^{n+1} + \Delta^+ B_j^n) R_j^{n+1} R_j^n]$$

$$+ \frac{1}{4k} [(R_j^{n+1} \Delta^+ B_j^{n+1})^2 + (R_j^n \Delta^+ B_j^n)^2 - (R_j^n \Delta^+ B_j^n)^2 - (R_j^{n-1} \Delta^+ B_j^{n-1})^2]$$

$$\frac{1}{2\lambda} R_j^n \delta^0_{R_j} \Delta^+ B_j^n^2$$

$$\frac{1}{2\lambda} R_j^n \delta^0_{R_j} \Delta^0 M_j^n^2$$

$$\frac{\delta^0_{R_j}}{2\lambda R_j^n} M_j^n^2 \Delta^0_{R_j}^2$$

$$\delta^0_{R_j} \frac{\Delta^0 Z_j^n}{4Z_j^n} = \frac{1}{4} \Delta^+ (Z_{j-1}^n \delta^0_{R_j} + Z_j^n \delta^0_{R_{j-1}}) \\ + \frac{1}{32} \frac{1}{k} \left[\left(\frac{Z_j^{n+1}}{R_j^{n+1}} + \frac{Z_j^n}{R_j^n} \right) \Delta^0 (R_j^{n+1^2} + R_j^{n^2}) - \left(\frac{Z_j^n}{R_j^n} + \frac{Z_j^{n-1}}{R_j^{n-1}} \right) \Delta^0 (R_j^{n^2} + R_j^{n+1^2}) \right]$$

$$- \frac{2jh}{R_j^n} \delta^0_{R_j} = \frac{jh}{2k} \left[\left(\frac{1}{R_j^{n+1}} \right)^2 + \left(\frac{1}{R_j^n} \right)^2 - \left(\frac{1}{R_j^n} \right)^2 - \left(\frac{1}{R_j^{n-1}} \right)^2 \right] \\ + \frac{jh}{2k} \left[2 \left(\frac{R_j^{n+1} - R_j^{n-1}}{R_j^n} \right) - \frac{R_j^{n+1^2} - R_j^{n-1^2}}{(R_j^{n+1} - R_j^{n-1})^2} \right]$$

$$\begin{aligned}
\frac{-1}{\lambda^2} R_j^n J_j^n \delta^0 \Delta^2 (R_j^n M_j^n) &= \frac{-1}{8\lambda^2} \delta^- [(\Delta^-(R_{j+1}^{n+1} M_{j+1}^{n+1}) - \Delta^-(R_{j+1}^n R_{j+1}^n))^2 \\
&+ (\Delta^-(R_j^{n+1} M_j^{n+1}) - \Delta^-(R_j^n M_j^n))^2] - \frac{1}{2\lambda^2} \Delta^- [(R_{j+1}^n M_{j+1}^n + R_j^n M_j^n) \\
&\cdot \Delta^- \delta^0 (R_{j+1}^n M_{j+1}^n)] + \frac{1}{8\lambda_k^2} [(\Delta^-(R_{j+1}^{n+1} M_{j+1}^{n+1}))^2 + (\Delta^-(R_{j+1}^n M_{j+1}^n))^2 \\
&+ (\Delta^-(R_j^{n+1} M_j^{n+1}))^2 + (\Delta^-(R_j^n M_j^n))^2 - (\Delta^-(R_{j+1}^n M_{j+1}^n))^2 \\
&- (\Delta^-(R_{j+1}^{n-1} M_{j+1}^{n-1}))^2 - (\Delta^-(R_j^n M_j^n))^2 - (\Delta^-(R_j^{n-1} M_j^{n-1}))^2]
\end{aligned}$$

$$\frac{1}{2\lambda} M_j^{n^2} \delta^0 \Delta^0 R_j^n$$

$$\begin{aligned}
\frac{1}{2\lambda} M_j^n \Delta^0 R_j^{n^2} \delta^0 M_j^n &= \frac{1}{16\lambda} \delta^- [M_j^{n+1} M_j^n (\Delta^0 R_j^{n+1^2} + \Delta^0 R_j^{n^2})] + \frac{M_j^n}{4\lambda} \delta^0 (M_j^n \Delta^0 R_j^{n^2}) \\
&+ \frac{1}{16\lambda_k} [M_j^{n+1} M_j^n (\Delta^0 R_j^{n+1^2} + \Delta^0 R_u^{n^2}) - M_j^n M_j^{n-1} (\Delta^0 R_j^{n^2} + \Delta^0 R_j^{n-1^2})] \\
&- \frac{1}{2\lambda} \frac{\delta^0 R_j^n}{R_j^n} M_j^{n^2} \Delta^0 R_j^{n^2}
\end{aligned}$$

$$- \frac{1}{\lambda^2} B_j^n \Delta^- \delta^0 (R_j^{n^2} \Delta^+ B_j^n)$$

$$\begin{aligned}
\frac{1}{2\lambda} B_j^n \delta^0 (B_j^n \Delta^- R_j^{n^2}) &= \frac{1}{8\lambda} \delta^- (B_j^{n+1} B_j^n \Delta^- (R_j^{n+1^2} + R_j^{n^2})) - \frac{1}{4\lambda} \delta^- (B_j^n B_j^{n+1}) \Delta^+ R_j^{n^2} \\
&+ \frac{1}{8\lambda_k} [B_j^{n+1} B_j^n \Delta^- (R_j^{n+1^2} + R_j^{n^2}) - B_j^n B_j^{n-1} (R_j^{n^2} + R_j^{n-1^2})].
\end{aligned}$$

Notice that we have substituted the fourth equation of (A2.1) directly into the sixth equation in the first group. Each term is multiplied by h and the first group is summed from $j = 0$ to $J-1$ while the second and third are summed from $j = 1$ to $J-1$. We note the following identities:

$$\begin{aligned}
& \frac{1}{2\lambda} (R^n \delta^0 R^n, \Delta^+ B^{n^2})_0 = - \frac{1}{2\lambda} (\Delta^-(R^n \delta^0 R^n), B^{n^2})_1 \\
& + \frac{1}{2\lambda} R_{J-1}^n B_J^{n^2} \delta^0 R_{J-1}^n - \frac{1}{2\lambda} R_0^n B_0^n \delta^0 R_n \\
& - \frac{1}{\lambda^2} (B_J^n \Delta^- \delta^0 (R^{n^2} \Delta^+ B^n))_1 = \frac{1}{\lambda^2} (\Delta^+ B_J^n \delta^0 (R^{n^2} \Delta^+ B^n))_0 \\
(A2.2) \quad & + \frac{1}{\lambda^2} B_J^n \delta^0 (R_{J-1}^{n^2} \Delta^+ B_{J-1}^n) + \frac{1}{\lambda^2} B_0^n \delta^0 (R_0^{n^2} \Delta^+ B_0^n) \\
& \frac{1}{2\lambda} (R^n \delta^0 R^n, \Delta^0 M^{n^2})_0 = - \frac{1}{2\lambda} (\Delta^0 (R^n \delta^0 R^n), M^{n^2})_1 \\
& + \frac{1}{4\lambda} (R_J^n \delta^0 R_J^n M_{J-1}^n + R_{J-1}^n \delta^0 R_{J-1}^n M_J^{n^2}) \\
& - \frac{1}{4\lambda} (R_0^n \delta^0 R_0^n M_1^n + R_1^n \delta^0 R_1^n M_0^{n^2}) .
\end{aligned}$$

We now add all terms and, remembering the definition (3.11) of E^n , we obtain

$$\begin{aligned}
& \frac{E^{n+1} - E^n}{k} + \frac{1}{2\lambda} R_{J-1}^n B_J^{n^2} \delta^0 R_{J-1}^n \\
& + \frac{1}{4\lambda} (R_J^n M_{J-1}^{n^2} \delta^0 R_J^n + R_{J-1}^n M_J^{n^2} \delta^0 R_{J-1}^n) \\
(A2.3) \quad & - \frac{1}{\lambda^2} B_J^n \delta^0 (R_{J-1}^{n^2} \Delta^+ B_{J-1}^n) + \frac{1}{2\lambda^2} (R_J^n M_J^n + R_{J-1}^n M_{J-1}^n \delta^0 \Delta^- (R_J^n M_J^n) \\
& + \frac{1}{4} \alpha^+ (Z_{J-1}^n \delta^0 R_J^n + Z_J^n \delta^0 R_{J-1}^n) = \frac{1}{\lambda} (\mathcal{B} + \mathcal{M}) + \frac{1}{\lambda^2} (\tilde{\mathcal{B}} + \tilde{\mathcal{M}}) + \tilde{\mathcal{P}}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{B} &= \frac{1}{4} (\Delta^- \delta^- (R^{n+1} R^n), B^{n^2})_1 - \frac{1}{8} (\delta^- (B^{n+1} B^n \Delta^- (R^{n+1^2} + R^{n^2})), 1)_1 \\
&+ \frac{1}{4} (\delta^- (B^n B^{n+1}), \Delta^- R^{n^2})_1, \\
\mathcal{M} &= \frac{1}{2} (M^{n^2}, \Delta^0 (R^n \delta^0 R^n))_1 - \frac{1}{2} (M^{n^2}, \Delta^0 \delta^0 R^{n^2})_1 \\
&- \frac{1}{8} (\delta^- (M^{n+1} M^n \Delta^0 (R^{n+1^2} + R^{n^2})), 1)_1 + \frac{1}{2} (M^n, \delta^0 (M^n \Delta^0 R^{n^2}))_1 \\
\tilde{\mathcal{B}} &= \frac{1}{4} (\delta^- (R^{n+1} \Delta^+ B^{n+1} - R^n \Delta^+ B^n)^2, 1)_1 - (\Delta^+ B^n, \delta^0 (R^{n^2} \Delta^+ B^n))_1 \\
&+ \frac{1}{2} (\Delta^+ B^n, \delta^- ((\Delta^+ B^{n+1} + \Delta^+ B^n) R^{n+1} R^n))_1 \\
\tilde{\mathcal{M}} &= \frac{1}{8} (\delta^- [(\Delta^+ (R^{n+1} M^{n+1}) - \Delta^+ (R^n M^n))^2], 1)_0 \\
&+ \frac{1}{8} (\delta^- [(\Delta^- (R^{n+1} M^{n+1}) - \Delta^- (R^n M^n))^2], 1)_1 \\
\tilde{\mathcal{P}} &= - \frac{\alpha^+}{2k} (Y, 2 \frac{R^{n+1} - R^{n-1}}{R^{n^2}} - \frac{R^{n+1^2} - R^{n-1^2}}{(R^{n+1} R^{n-1})^2})
\end{aligned}$$

and $Y_j = jh$.

There remains only to show that $\mathcal{B}$, $\mathcal{M}$, $\tilde{\mathcal{B}}$, $\tilde{\mathcal{M}}$, and $\tilde{\mathcal{P}}$ are $O(k^2)$ to arrive at (3.12). This is, however, easily done, for by simple manipulation it is found that these quantities can be written as

$$\begin{aligned}
\mathcal{B} &\equiv -\frac{k^2}{8} \{ (B^{n^2}, \Delta^- \delta^- (\delta^- R^{n+1})^2)_1 + (B^n, \delta^- [(\delta^- B^{n+1})(\Delta^- \delta^- R^{n+1})^2])_1 \} \\
\mathcal{M} &\equiv -\frac{k^2}{8} \{ (M^{n^2}, \Delta^0 \delta^- (\delta^- R^{n+1})^2)_1 + (M^n, \delta^- [(\delta^- M^{n+1})(\Delta^0 \delta^- R^{n+1})^2])_1 \} \\
\tilde{\mathcal{B}} &\equiv -\frac{k^2}{4} \{ ((\Delta^+ B^n)^2, \delta^- (\delta^- R^{n+1})^2)_1 - (\delta^- (\delta^- \Delta^+ B^{n+1})^2, R^{n^2})_1 \\
&\quad - ((\delta^- \Delta^+ B^{n+1})^2, \delta^- R^{n+1})_1 + ((\delta^- \Delta^+ B^n)^2, \delta^- R^{n^2})_1 \} \\
\tilde{\mathcal{M}} &\equiv \frac{k^2}{8} \{ \delta^- (\delta^- \Delta^+ (R^{n+1} M^{n+1}))^2, 1 \}_0 + (\delta^- (\delta^- \Delta^- (R^{n+1} M^{n+1}))^2, 1)_1 \} \\
\tilde{\mathcal{P}} &\equiv -\frac{\alpha + k^2}{2} \left(-\frac{Y}{R^{n-1} R^n R^{n+1}} \frac{1}{R^{n+1} R^n} , 2R^{n+1} R^{n-1} (\delta^+ R^n)(\delta^+ R^{n-1}) \right. \\
&\quad \left. [\delta^+ R^n + \delta^+ R^{n-1} - R^n \delta^2 R^n] - R^n [R^{n^2} \delta^- (\delta^+ R^n)^2 \right. \\
&\quad \left. - (\delta^+ R^{n-1})^2 (\delta^+ R^n)^2 - (\delta^+ R^{n^2}) (\delta^+ R^{n-1})^2 \right]_1 .
\end{aligned}$$

From (A2.4) we see that if R , B and M possess sufficient bounded derivatives then $\mathcal{B}$, $\mathcal{M}$, $\tilde{\mathcal{B}}$, $\tilde{\mathcal{M}}$, and $\tilde{\mathcal{P}}$ will all be $O(k^2)$.

APPENDIX 3. Boundary Conditions

Because of the mixed hyperbolic-parabolic nature of the system (2.48) the number of boundary conditions one would expect to be necessary to determine uniquely the solution is given by counting characteristics taking into account infinite characteristic speeds. The inclusion of pressure contributes an inward pointing characteristic with slope $dY/dT = -\sqrt{c_+^2 + c_-^2}$. At the inner boundary $R = 0$ this speed is seen to vanish so the presence of pressure necessitates specifying one additional boundary condition at the outer edge.

From a physical viewpoint this condition should reflect the fact that as the plasma moves inward a large pressure gradient develops which smooths out the edge and prevents the formation of a vacuum. One way to handle this is to impose the condition that the gas "sticks" to the wall. Numerically, this means that the value of R at the last mesh point remains fixed at its initial value. That is

$$(A3.1) \quad R_J^n = R_J^0 .$$

There are, however, some serious drawbacks to this condition. One is that as the density, and hence the pressure, becomes small the problem becomes badly behaved at the boundary since the existence of this condition depended on the presence of pressure. Secondly, the large gradients in density which develop as the plasma is driven inward give rise to numerical

instabilities which can only be eliminated by taking a prohibitively small ratio of k/h in the difference method. For these reasons we do not adopt this condition. Instead, we assume that the ion pressure effects at the boundary can be neglected compared to the force produced by the compressing magnetic fields. We thus postulate

$$(A3.2) \quad (RP_{11}^+)_{\text{Y}} - R_{\text{Y}}P_{22}^+ = \left(-\frac{\alpha^+}{2YR_{\text{Y}}^3}\right)_{\text{Y}} - \frac{2\alpha_{\text{Y}}^+}{R^3} = 0$$

at the boundary. A sufficient condition for (A3.2), which we use as the boundary condition when $\alpha^+ \neq 0$ is

$$(A3.3) \quad N_{\text{Y}} = 0 .$$

The electron pressure, by virtue of its dependence on the magnetic fields still contributes a force at the boundary. However, as we see from the characteristic equation, the two pressures contribute only one additional characteristic so no new boundary condition need be imposed when the electron pressure is added. Since $N = 2/(R^2)_{\text{Y}}$, (A3.3) can be written

$$(A3.4) \quad R_{\text{J}}^{n^2} - 2R_{\text{J}-1}^{n^2} + R_{\text{J}-2}^n = 0 .$$

A heuristic indication that (A3.4) is a reasonable boundary condition to impose is given by examining the pressure-free problem. In this case, because of the explicit presence of the variable R , it is not possible to solve for R at the last mesh point while maintaining the desired

accuracy of the difference method. This arises because of the centering of the discrete variables and is absent in a plane geometry. See Morton (15). To determine R_J^n in our difference scheme we extrapolate to obtain

$$(A3.5) \quad R_J^{n^2} = 2 R_{J-1}^{n^2} - R_{J-2}^{n^2} + O(h^2) .$$

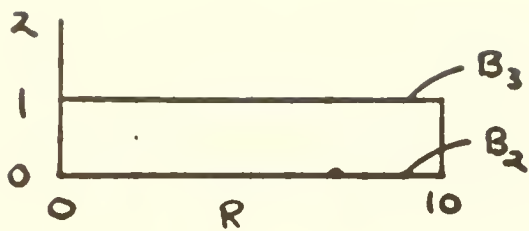
Better accuracy is obtained by extrapolating on R^2 rather than R because R behaves like $\sqrt{Y}$. However, (A3.5) is exactly condition (A3.4) we imposed for the problem in which pressure was present. Thus, the condition (A3.4) plays two roles although, as a practical matter, its roles are indistinguishable. One advantage to this is that by not introducing new conditions into the problem we can easily observe the effects of pressure on the disturbances which develop in the fluid. Of course, at the same time we lose all information about the boundary layer at the edge.

Morton (16) uses a similar boundary condition at the plasma-vacuum interface in the plane case.

BIBLIOGRAPHY

1. Adlam, J. H. and Allen, J. E., "The structure of steady collision-free hydromagnetic waves." *Phil. Mag.* 3, 448 (1958).
2. Auer, P. L., Hurwitz, Jr., H. and Kilb, R. W., "Large amplitude magnetic compression of a collision-free plasma." *Phys. Fluids* 5, 298 (1962).
3. Banos, Jr., A. and Vernon, A. R., "Large amplitude waves in a collision-free plasma. I. Single pulses with isotropic pressure." *Nuovo Cimento* 15, 269 (1960).
4. Chew, G. F., Goldberger, M. L., and Low, F. E., "The Boltzman equation and the one-fluid hydromagnetic equation in the absence of particle collisions." *Proc. Roy. Soc. Ser. A*, 236, 112 (1956).
5. Friedrichs, K. O. and Kranzer, H., "Nonlinear wave motion." NYO-6486-VIII, Courant Inst. Math. Sci., New York Univ. (1958).
6. Gardner, C. S. and Morikawa, G. K., "The effect of temperature on the width of a small amplitude, solitary wave in a collision-free plasma." *Comm. Pure Appl. Math.* 18, 35 (1965).
7. Gardner, C. S., Goertzel, H., Grad, H., Morawetz, C. S., Rose, M. H. and Rubin, H., "Hydromagnetic shock waves in high-temperature plasmas." Proc. 2nd U.N. Intl. Conf. on Peaceful Uses of Atomic Energy, Vol. 31, 230 (1958).
8. Grad, H., "Microscopic and macroscopic models in plasma physics." MF-19, Courant Inst. Math. Sci., New York Univ. (1961).
9. Grad, H., "The guiding center plasma." *Proc. Symp. Appl. Math.*, Vol. 18, 162 (1967).
10. Hain, K., Hain, G., Roberts, K. V., Roberts, S. J. and Köppendörfer, W., "Fully ionized pinch collapse." *Z. Naturforsch.* 15a, 1039 (1960).
11. Hain, K., Lüst, R., Schlüter, A., "Hydromagnetic waves of finite amplitude with isotropic and non-isotropic pressure perpendicular to a magnetic field." *Revs. Mod. Phys.* 32, 967 (1960).

12. Kilb, R. W., "Theta pinch shock implosion calculations by model simulation of collisionless ion Vlasov equation." 67-C-166, General Electric Research and Development Center, Schenectady, N.Y. (1967).
13. Morawetz, C. S., "Collisionless shocks and solitary waves." NYO-1480-12, Courant Inst. Math. Sci., New York Univ. (1964).
14. Morawetz, C. S. "Magnetohydrodynamic shock structure without collisions." Phys. Fluids 4, 988 (1961).
15. Morton, K. W. "The development of compression waves in an adiabatic two-fluid model of a collision-free plasma." NYO-9763, Courant Inst. Math. Sci., New York Univ. (1961).
16. Morton, K. W., "Finite amplitude compression waves in a collision-free plasma." Phys. Fluids 7, 1800 (1964).
17. Oliphant, Jr., T. A., "Computer simulation of the theta pinch." LAMS-2944, Los Alamos (1963).
18. Richtmyer, R. D. Difference Methods for Initial Value Problems. New York: Interscience (1957).
19. Rosenbluth, M. "Infinite conductivity theory of the pinch." LA-1850, Los Alamos (1955).
20. Rossow, V. J. "Magnetic-compression waves in collisionless plasmas -- oblique ambient magnetic field." Phys. Fluids 10, 1056 (1967).



**INITIAL MAGNETIC
FIELDS**

$$B_2 = 0$$

$$B_3 = \begin{cases} 1 + \frac{T}{10} & 0 \leq T \leq 10 \\ 2 & 10 \leq T \end{cases}$$

FIELDS AT BOUNDARY

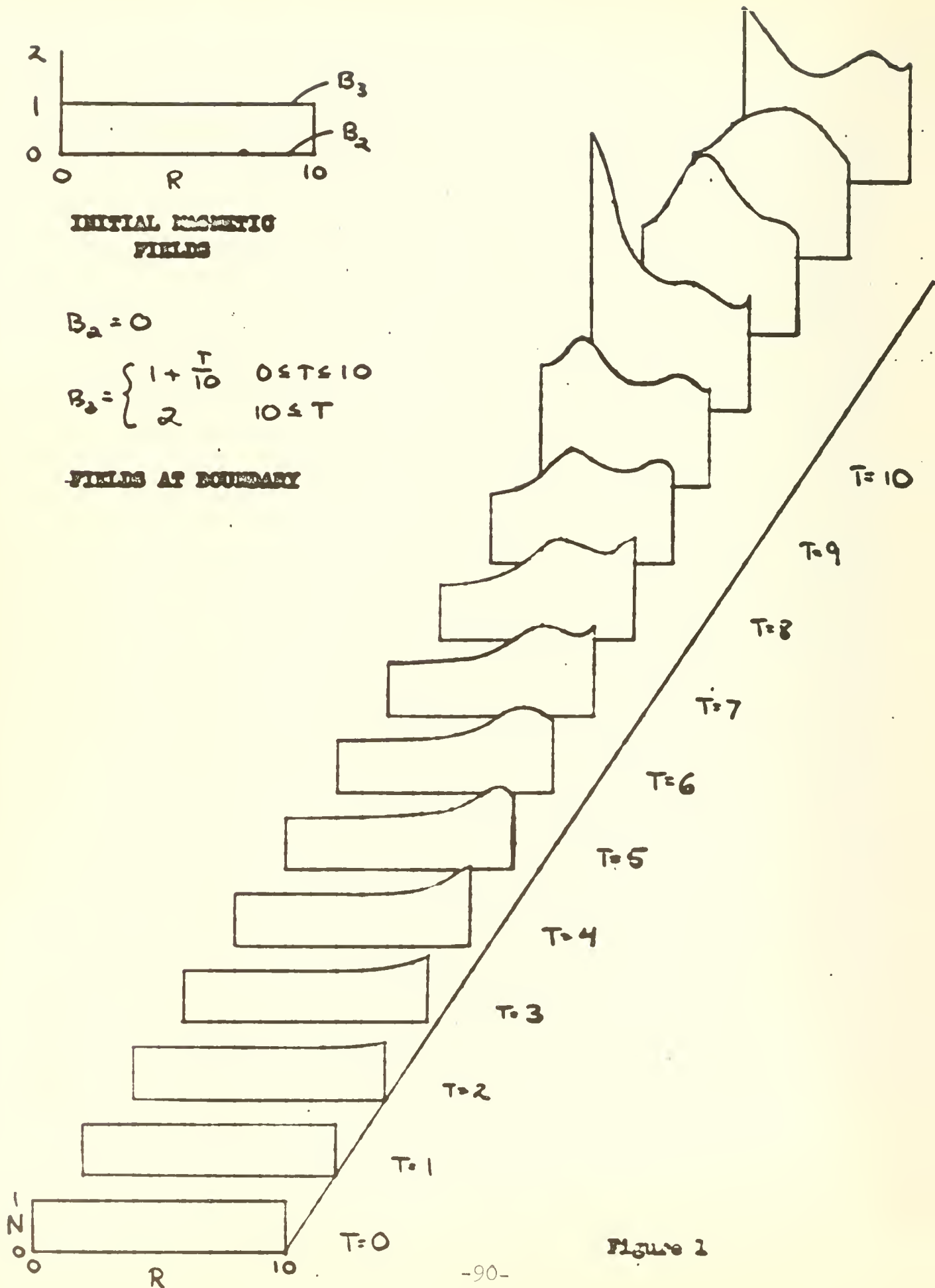
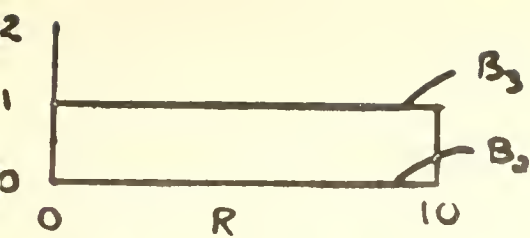


Figure 1



INITIAL MAGNETIC
FIELDS

$$B_a = 0$$

$$B_3 = \begin{cases} 1 + \frac{T}{5} & 0 \leq T \leq 7 \\ 2 & 7 \leq T \end{cases}$$

FIELDS AT BOUNDARY

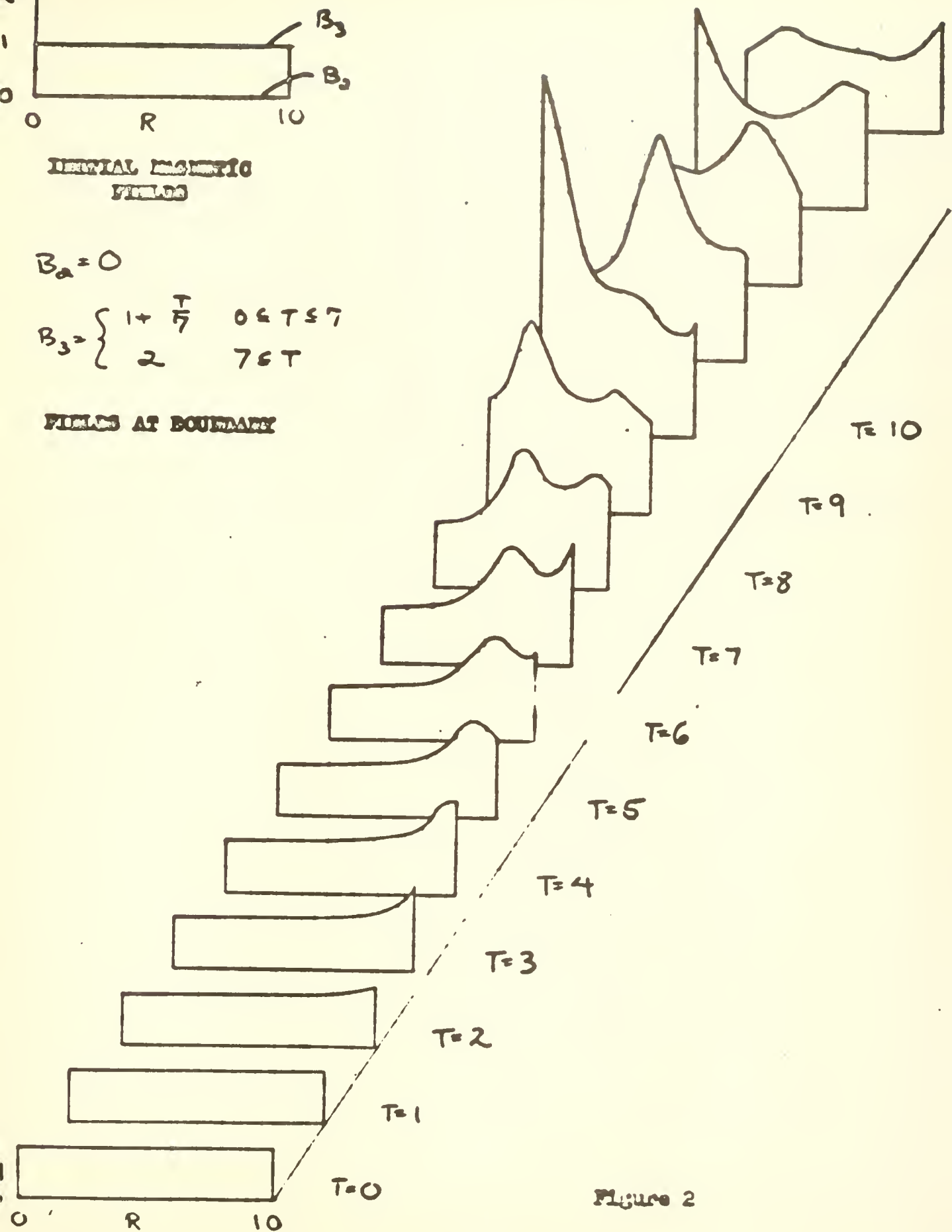
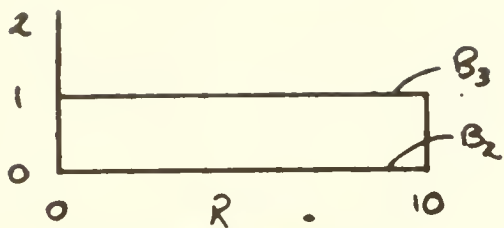


Figure 2

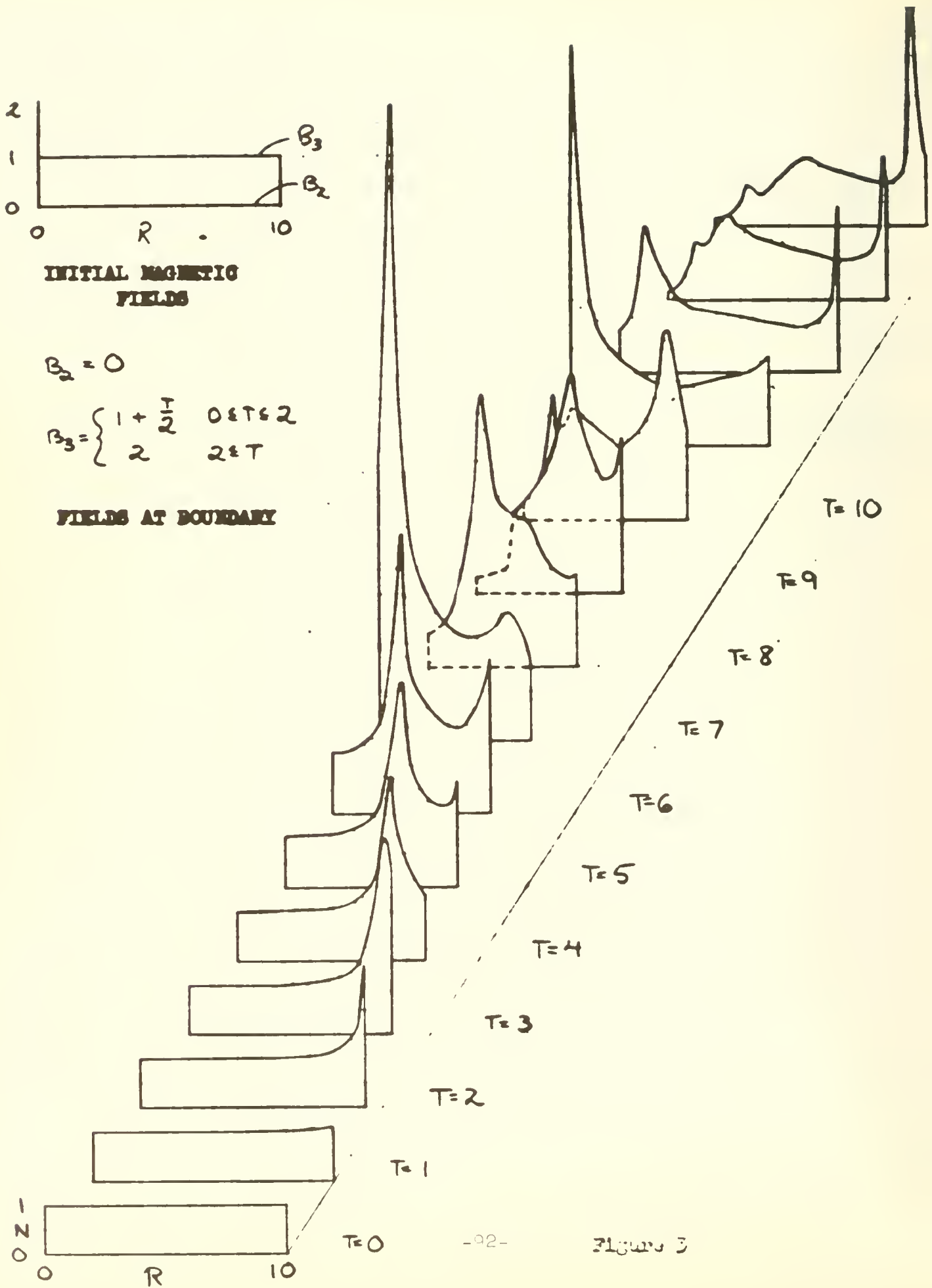


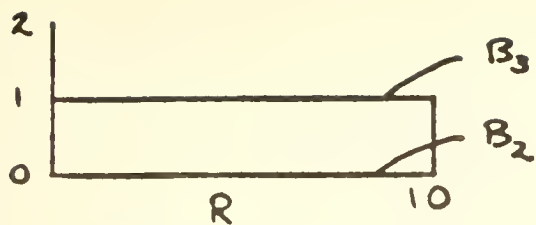
INITIAL MAGNETIC
FIELDS

$$B_2 = 0$$

$$B_3 = \begin{cases} 1 + \frac{T}{2} & 0 \leq T \leq 2 \\ 2 & 2 \leq T \end{cases}$$

FIELDS AT BOUNDARY





INITIAL MAGNETIC
FIELDS

$$B_2 = 0$$

$$B_3 = \begin{cases} 1 + \frac{T}{7} & 0 \leq T \leq 7 \\ 2 & 7 \leq T \end{cases}$$

FIELDS AT BOUNDARY

$$\alpha^+ = \alpha^- = .04$$

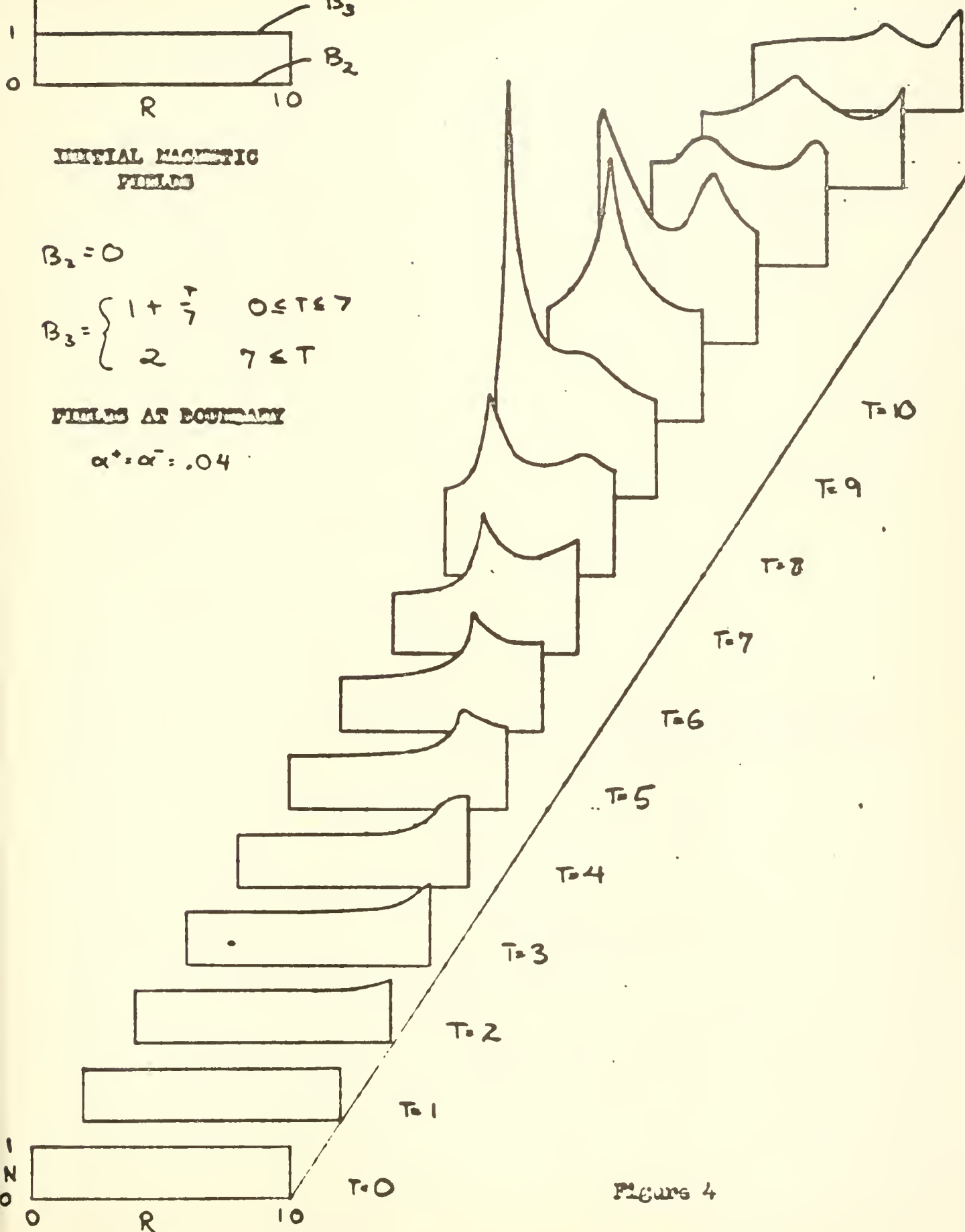
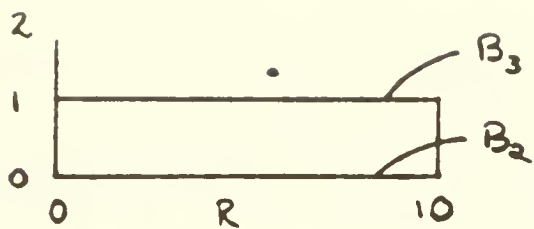


Figure 4



INITIAL MAGNETIC FIELDS

$$B_2 = \begin{cases} .06T & 0 \leq T \leq 10 \\ .6 & 10 \leq T \end{cases}$$

$$B_{\theta y} = 0$$

FIELDS AT BOUNDARY

$$\alpha^+ = \alpha^- = .001$$

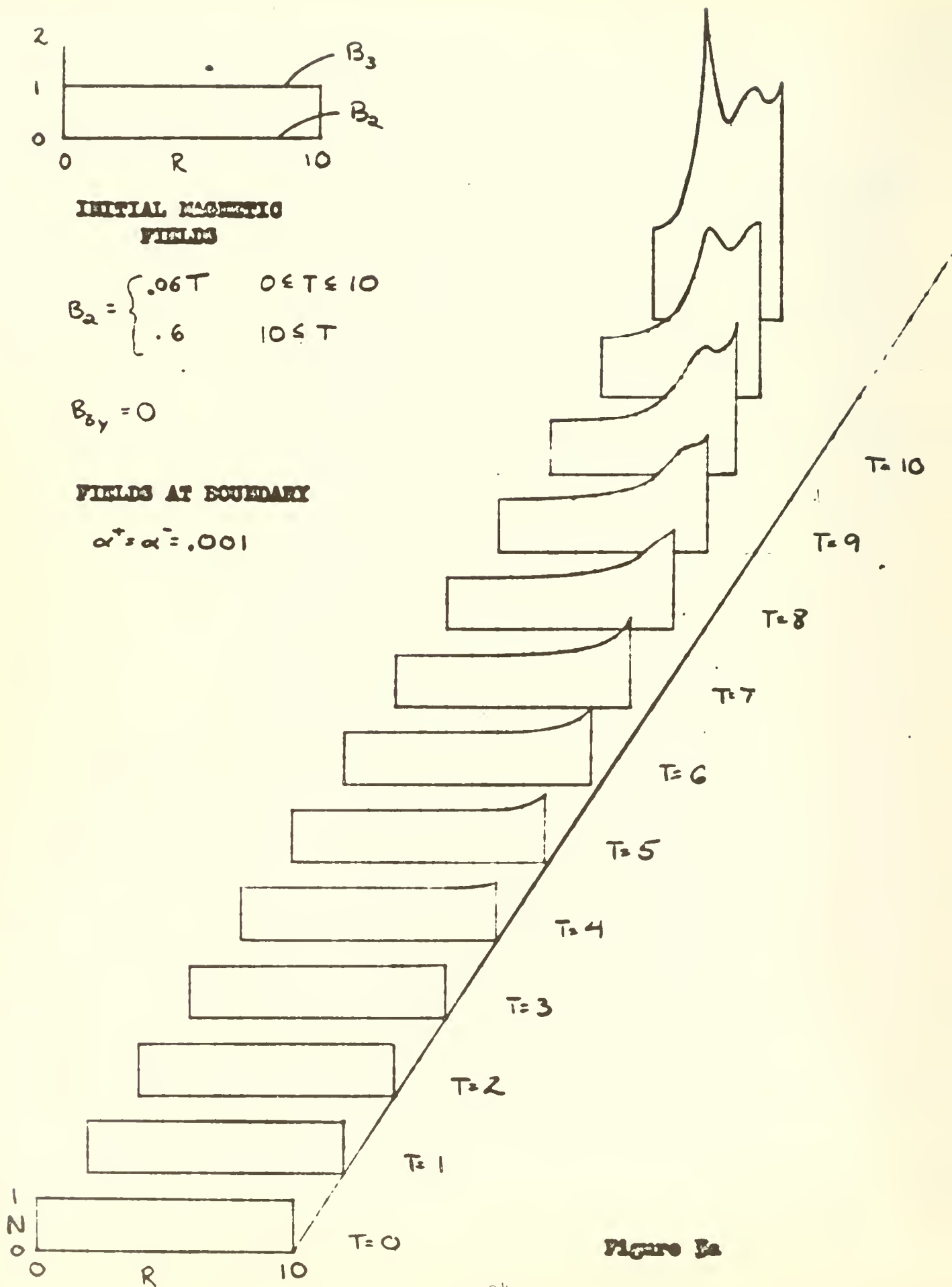
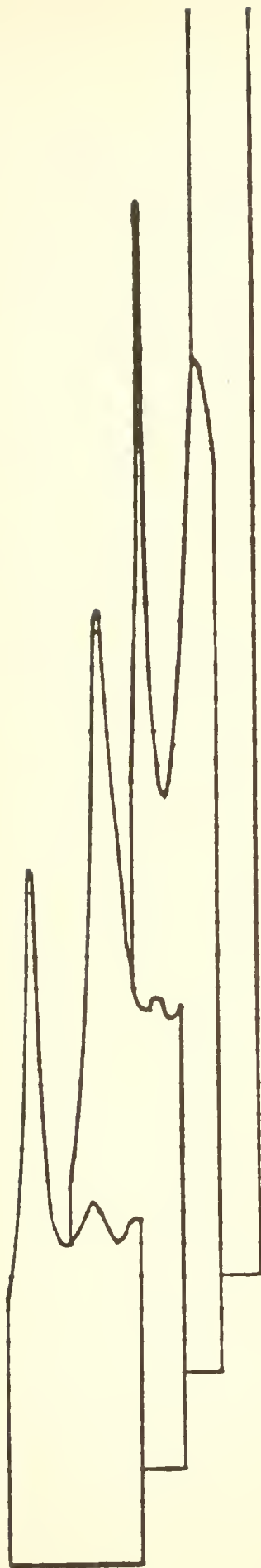


Figure 1a



DENSITY PROFILE FOR $T=14$
AT REDUCED DENSITY SCALE

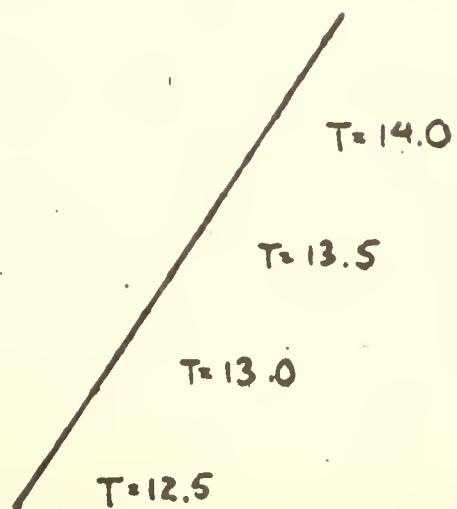
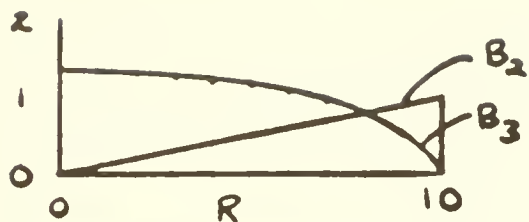


Figure 3b



INITIAL MAGNETIC
FIELDS

$$B_2 = \begin{cases} 1 + \frac{T}{10} & 0 \leq T \leq 10 \\ 2 & 10 \leq T \end{cases}$$

$$B_3 = 0$$

FIELDS AT BOUNDARY

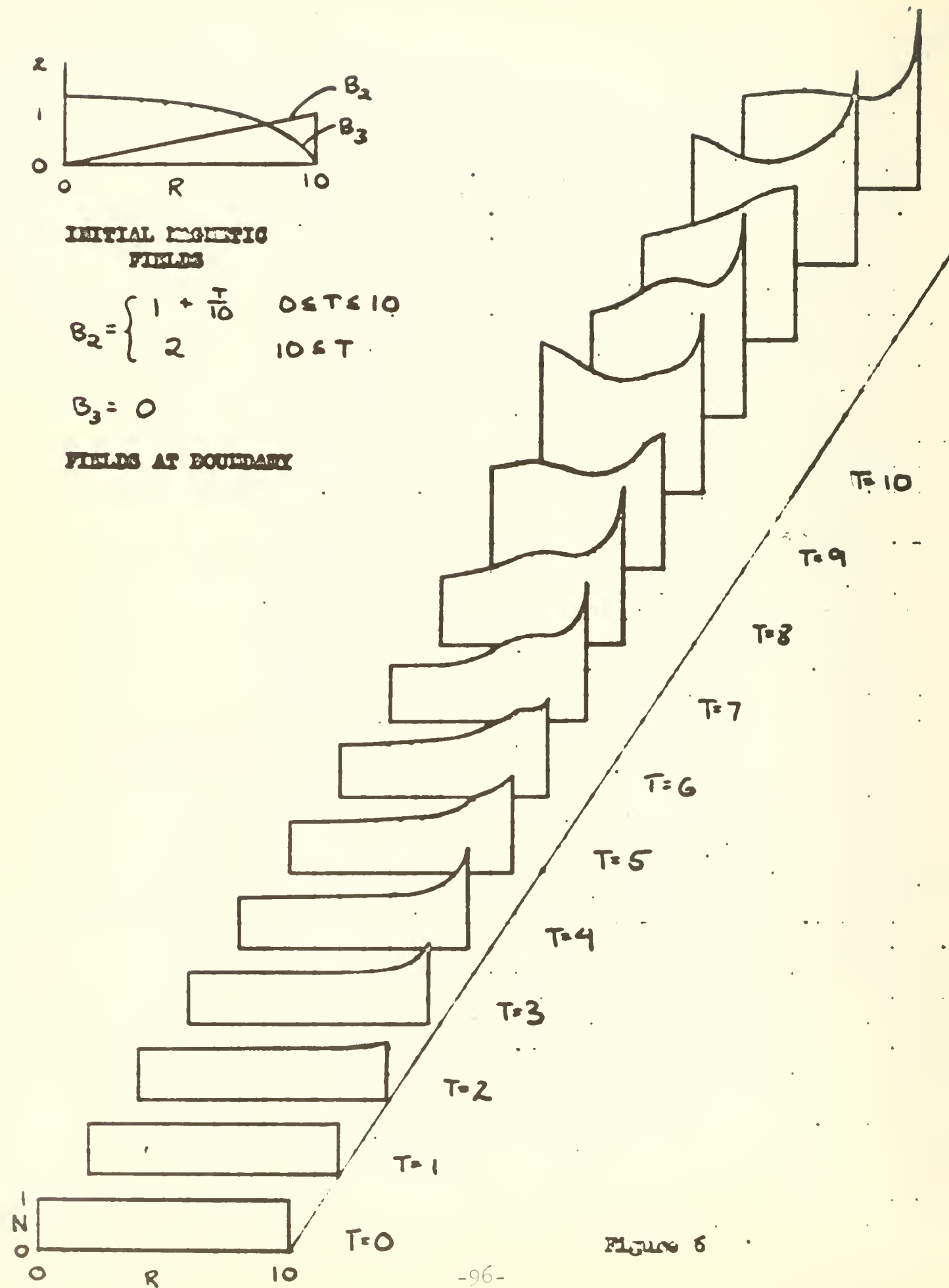
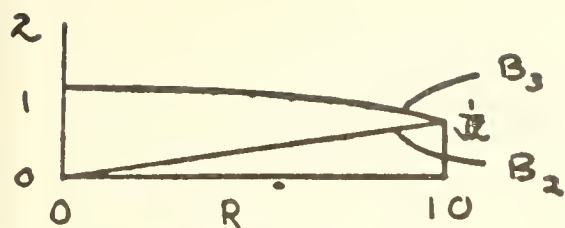


Figure 6



INITIAL MAGNETIC
FIELDS

$$B_2 = \begin{cases} \sqrt{\left(1 + \frac{T}{10}\right)^2 - \frac{1}{2}} & 0 \leq T \leq 10 \\ \sqrt{\frac{1}{2}} & 10 \leq T \end{cases}$$

$$B_2 = \frac{1}{\sqrt{2}}$$

FIELDS AT BOUNDARY

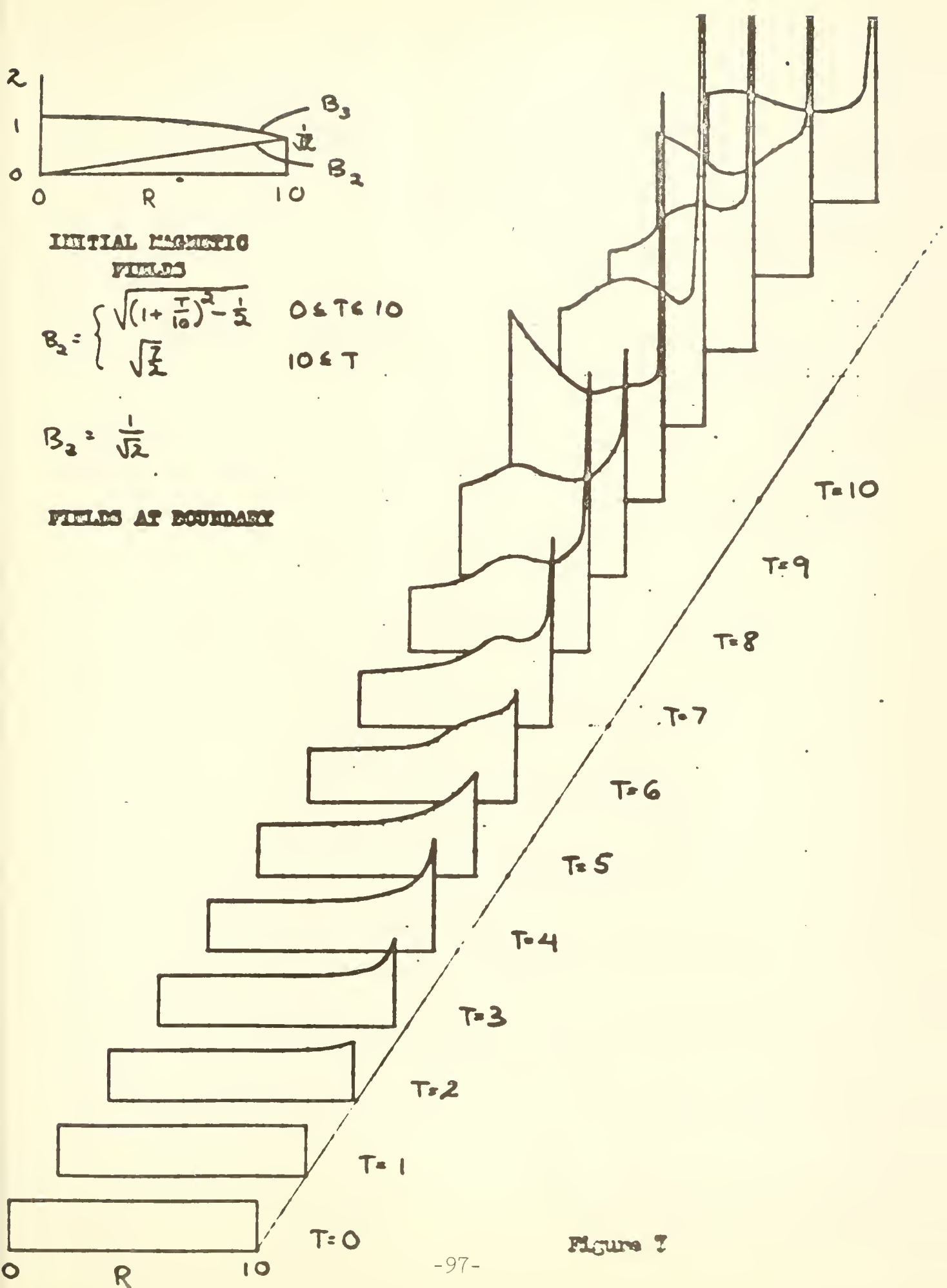
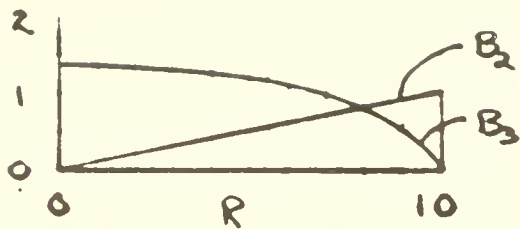


Figure 7

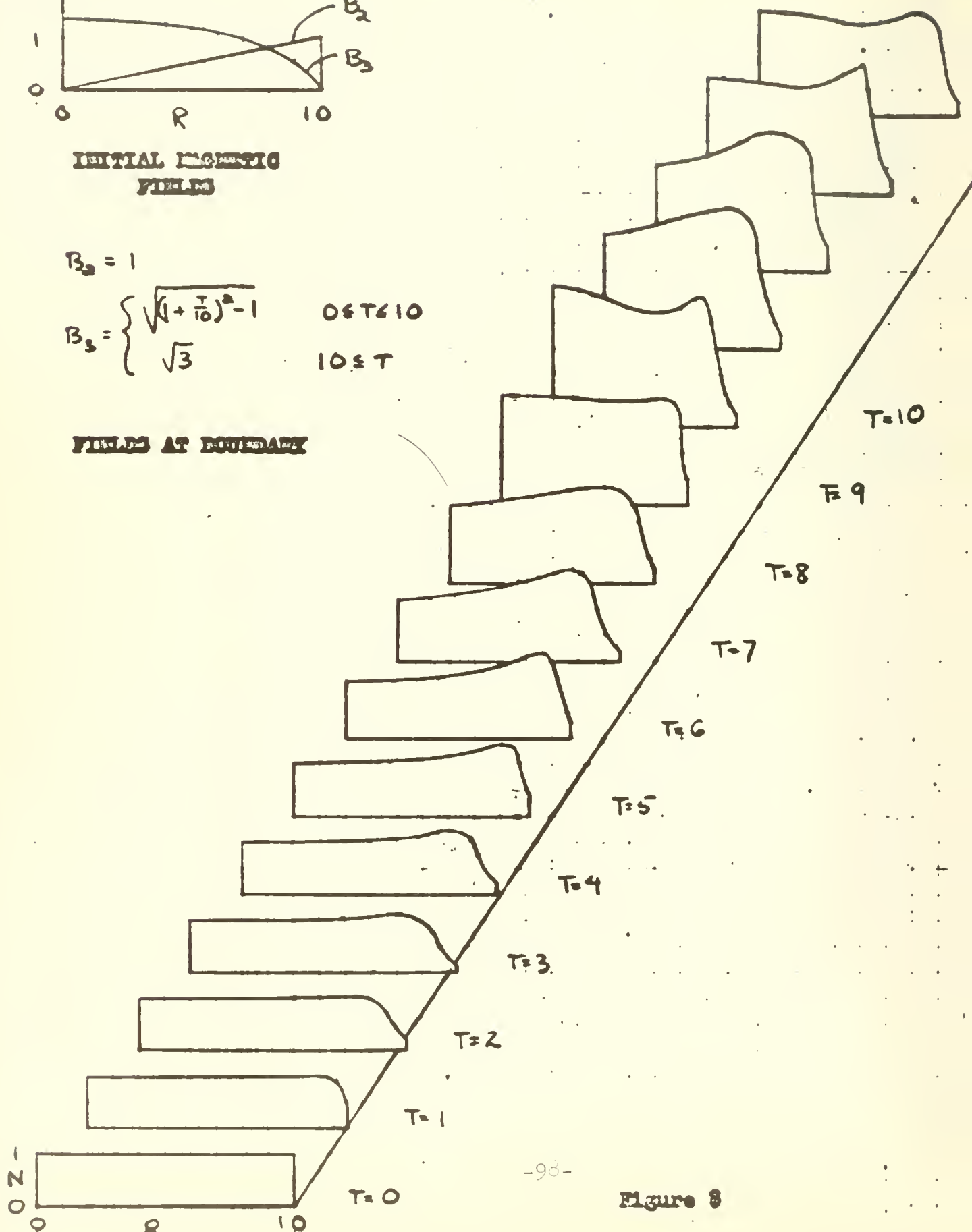


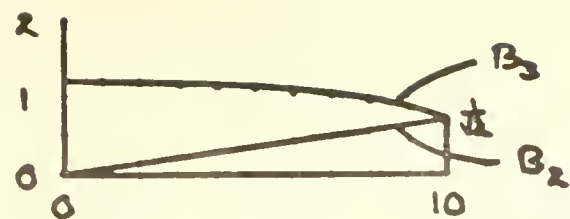
**INITIAL MAGNETIC
FIELDS**

$$B_2 = 1$$

$$B_3 = \begin{cases} \sqrt{(1 + \frac{T}{10})^2 - 1} & 0 \leq T < 10 \\ \sqrt{3} & 10 \leq T \end{cases}$$

FIELDS AT BOUNDARY





INITIAL MAGNETIC
FIELDS

$$B_2 = \frac{1}{\sqrt{2}}$$

$$B_2 = \begin{cases} \sqrt{(1+\frac{T}{T_0})^2 - \frac{1}{2}} & 0 \leq T \leq 10 \\ \frac{1}{\sqrt{2}} & 10 \leq T \end{cases}$$

FIELDS AT BOUNDARY

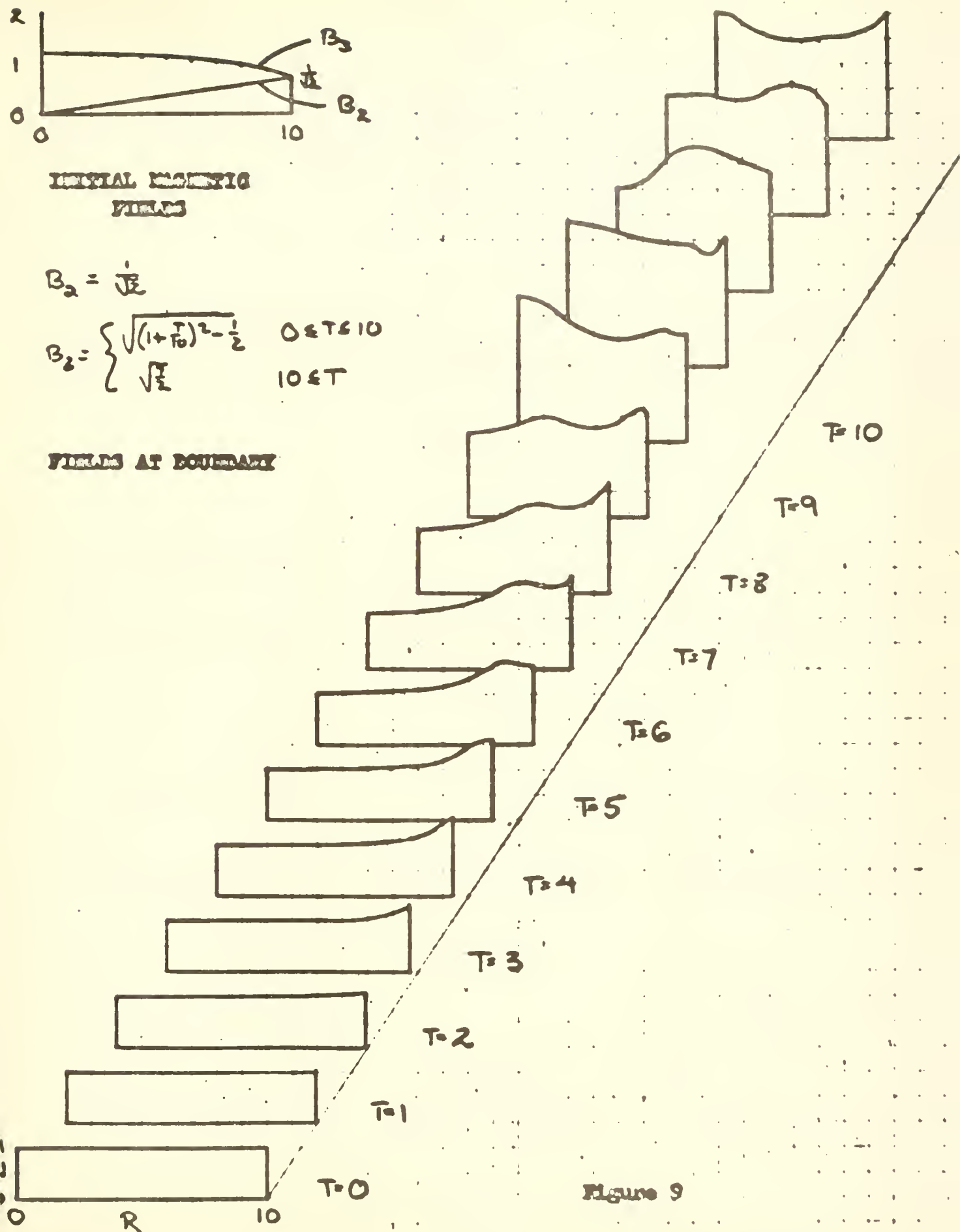
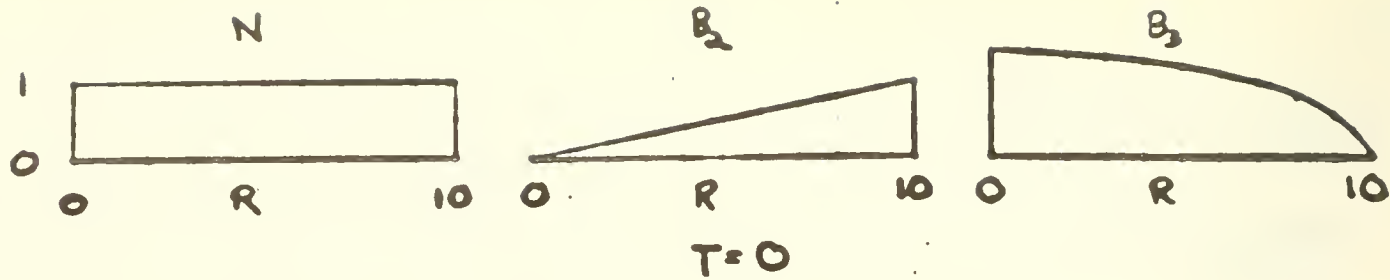
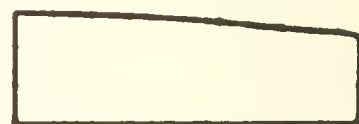


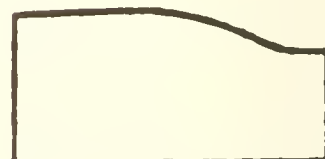
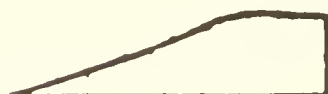
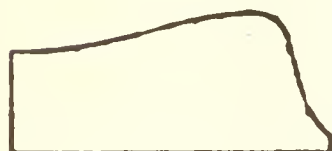
Figure 9



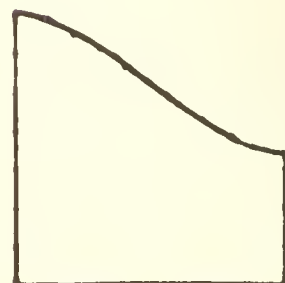
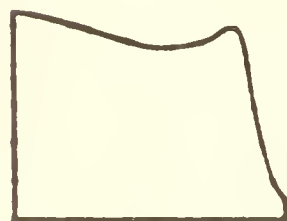
$T=2.5$



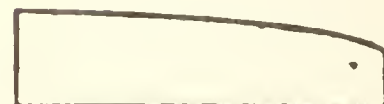
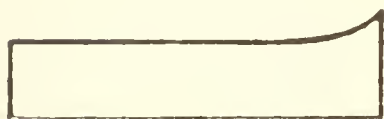
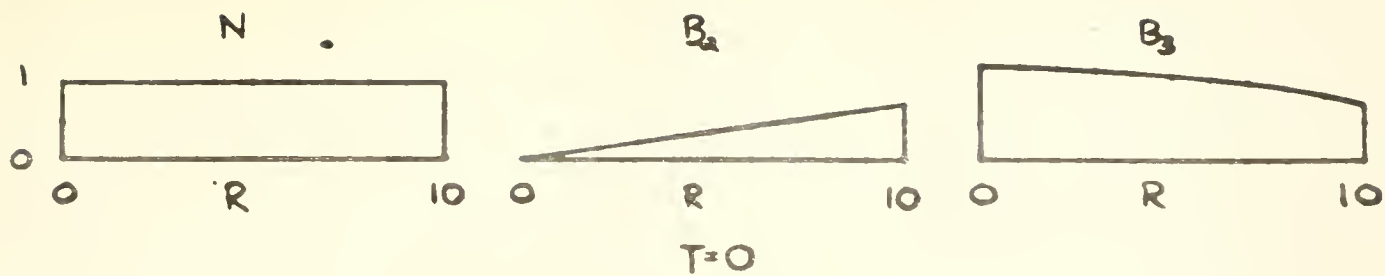
$T=5$



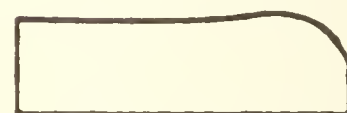
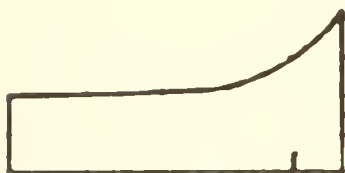
$T=7.5$



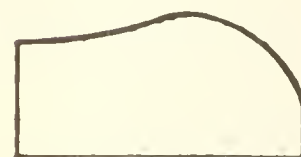
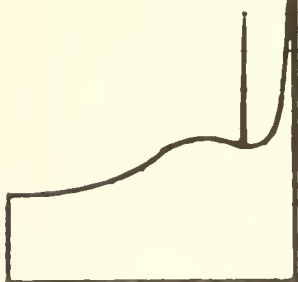
$T=10$



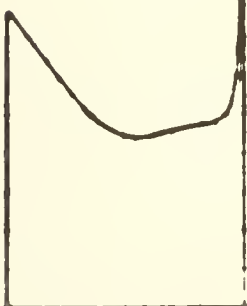
$T=2.5$



$T=5$



$T=7.5$



$T=10$

Figure 11

OCT 10 1969
Date Due

	MAY 1 1972			
Thd				
Gov				
Und				
per				
A.				
B.				
As				
of				
cor				
suc				
plc				
emp				
sen				
mat				
wit				
suc				

Demco 38-297

NYU
NYO-
1480-112 c.2

NYU
NYO- Marder c.2

NYU
NYO-
1480-112 Marder c.2
AUTHOR
TITLE
Compression of an axially
symmetric plasma with non-
isotropic pressure.

DATE DUE

BORROWER'S NAME

MAY 12 1972

Toni Wolfe

**N.Y.U. Courant Institute of
Mathematical Sciences**

251 Mercer St.
New York, N. Y. 10012

